# Functional series

## The concept of functional series

If it is

$$f_1(x),\ f_2(x), \ldots, f_n(x), \ldots$$

a sequence of real functions defined on the interval $I$. Then by an infinite functional series we mean a symbol

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + \ldots + f_n(x) + \ldots$$

## Domain of convergence

The set of numbers $x \in I$ for which the corresponding numerical series of functional values converges forms the so-called **convergence domain.**

### Examples

We determine the domain of convergence of functional series:

a) $\sum_{n=1}^{\infty} x^{n-1}$.

b) $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot n \cdot x^{n-1}$.

c) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$.

d) $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot 10^n \cdot x^n$.

e) $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

#### Solution a)

a) This is a geometric series with the quotient $q = x$. We know that geom. the series converges if and only if $|q| < 1$. holds. This means that our series converges if and only if $|x| < 1$, i.e. if $x \in (-1, 1)$.

#### Solution b)

Let's examine the abs. convergence:

$$\lim_n |\frac{a_{n+1}}{a_n}| = \lim_n |\frac{(n+1) \cdot x^n}{n \cdot x^{n-1}}| = \lim_n \frac{n+1}{n} \cdot |x| = |x| \cdot \lim_n \frac{n+1}{n} = |x| \cdot 1 = |x|.$$

It follows that a number of abs. converges if $|x| < 1$.

In the case where $|x| \geq 1$ is valid, the series diverges because the necessary condition of convergence is not met.

## Solution c)

c) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$.

Again, we will use the proportion criterion to examine the abs. convergence:

$$L = \lim_n |\frac{a_{n+1}}{a_n}| = \lim_n \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = |x| \cdot \lim_n \frac{1}{n+1} = |x| \cdot 0 = 0.$$

The limit of $L < 1$ and is not dependent on $x$. Hence the series converges for every $x$.

## Solution d)

$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot 10^n \cdot x^n$.

Let's use the so-called square root criterion:

$$L = \lim_n \sqrt[n]{10^n \cdot |x|^n} = 10 \cdot |x|.$$

$$L < 1 \iff x \in (-\frac{1}{10}, \frac{1}{10}).$$

For $x \notin (-\frac{1}{10}, \frac{1}{10})$ the series diverges.

## Solution e)

$\sum_{n=1}^{\infty} \frac{x^n}{n}$.

We use the square root criterion:

$$L = \lim_n \sqrt[n]{\frac{|x|^n}{n}} = |x| \cdot \lim_n \sqrt[n]{\frac{1}{n}} = |x|.$$

A sample limit was used here: $\lim_n \sqrt[n]{n} = 1$.

With a series of abs. converges if $x \in (-1, 1)$. Let's further examine the convergence for $x = \pm 1$.

Flight $x = -1$ :

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}.$$

As a consequence of the Leibniz criterion, this (alternating) series then converges.

Flight $x = 1$ :

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

So the domain of convergence is a semi-closed interval: $\langle -1, 1)$.

# Uniform convergence of functional series

The function series $\sum_n f_n(x)$ converges on the interval $I$ **uniformly to the function S(x)** if for every $\varepsilon > 0$ there exists a $N_\varepsilon \in \mathbb{N}$, such that for all $x \in I$ and for all $n > N$

$$|S(x) - s_n(x)| < \varepsilon.$$

Here for all $x \in I,\ n \in \mathbb{N}$ is

$$s_n(x) = \sum_{k=1}^{n} f_n(x).$$

We say that the series $\sum_{n=1}^{\infty} f_n(x)$ on the interval $I$ **converges pointwise to the function** $S(x)$, if for every $x \in I$ and for every $\varepsilon > 0$ there exists a $N_{\varepsilon,x} \in \mathbb{N}$ such that for every $n > N_{\varepsilon,x}$

$$|S(x) - s_n(x)| < \varepsilon.$$

**Example.** Let's examine the convergence of the series on the interval $I = \langle 0, 1)$:

$$\sum_{n=1}^{\infty} x^n.$$

**Solution.** First, let's examine the so-called point convergence. For $x \in I$ we have

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}, \quad \text{pro } |x| < 1.$$

Here is $S(x) = \frac{x}{1-x}$. And for the given series to converge pointwise to $S(x)$ it must hold:

$$\forall x \in I \quad \forall \varepsilon > 0 \quad \exists N_{\varepsilon,x} \in \mathbb{N}: \quad |S(x) - s_n(x)| < \varepsilon, \quad \text{pro } n > N_{\varepsilon,x}.$$

Where applicable:

$$s_n(x) = \sum_{k=1}^{n} x^k = \frac{x(1-x^n)}{1-x} \quad \text{pro } |x| < 1.$$

So it must apply:

$$|\frac{x}{1-x} - \frac{x(1-x^n)}{1-x}| < \varepsilon, \quad \text{pro } n > N_{\varepsilon,x}.$$

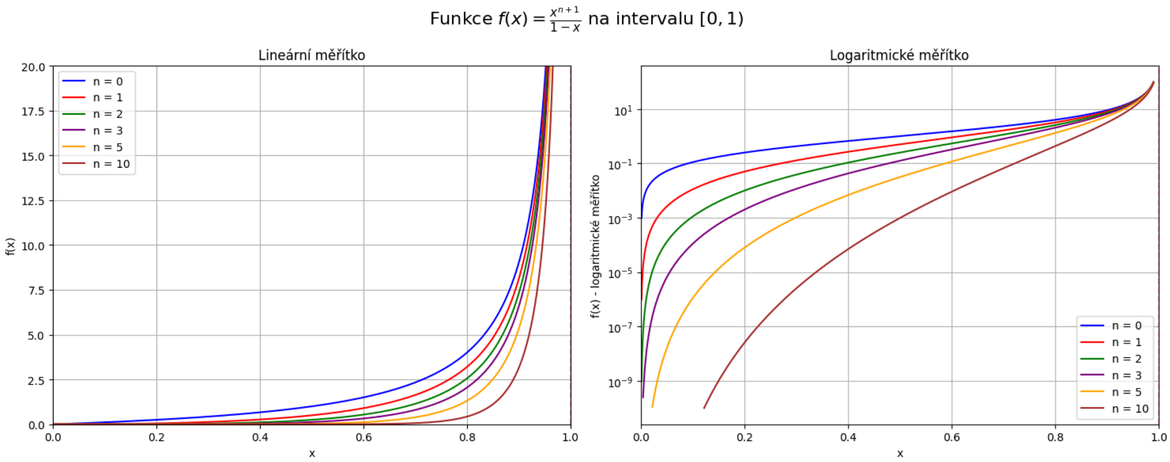
After simplifying, we get:

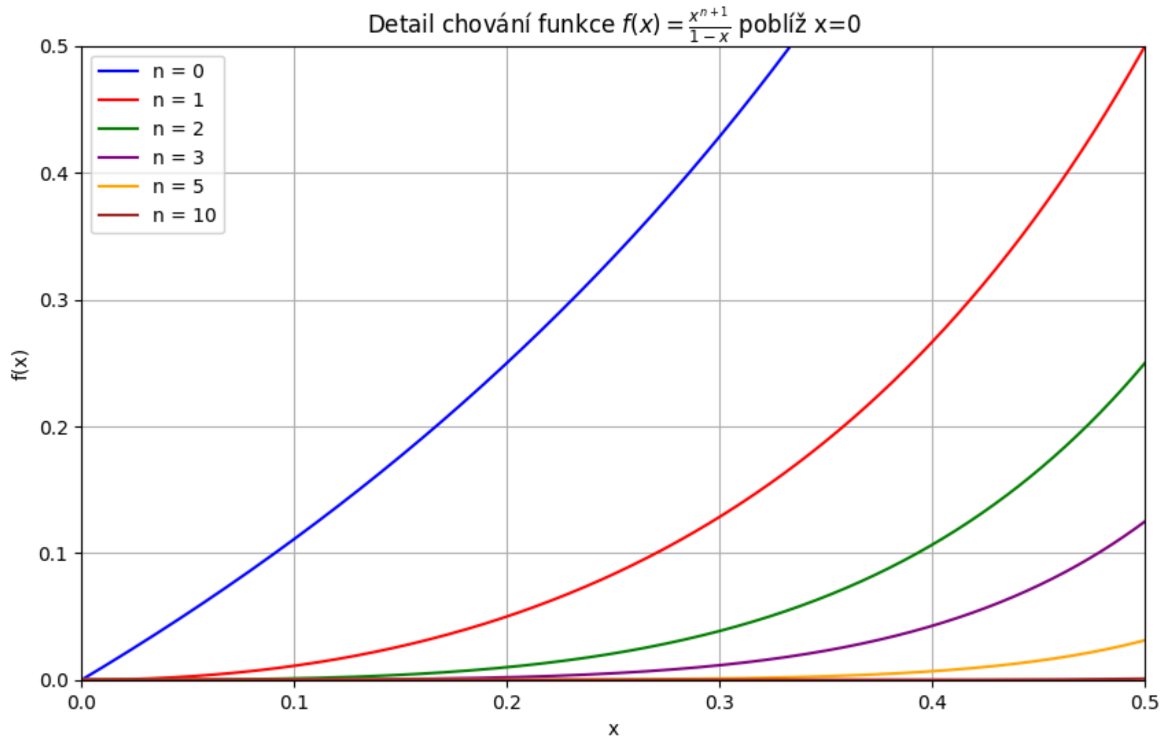
$$\frac{x^{n+1}}{1-x} < \varepsilon, \quad \text{pro } n > N_{\varepsilon,x}.$$

In [1]:

```
import numpy as np
import matplotlib.pyplot as plt

# Definice funkce
def f(x, n):
    return (x**(n+1)) / (1-x)

# Vytvoření více grafů pro různé pohledy
fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(15, 6))

# Generování hodnot x (vyhneme se přesně x=1, kde funkce není definována)
x = np.linspace(0, 0.99, 1000)

# Různé hodnoty n pro vykreslení
n_values = [0, 1, 2, 3, 5, 10]
colors = ['blue', 'red', 'green', 'purple', 'orange', 'brown']

# Graf 1: Lineární měřítko
for i, n in enumerate(n_values):
    y = f(x, n)
    ax1.plot(x, y, label=f'n = {n}', color=colors[i])

ax1.set_xlim(0, 1)
ax1.set_ylim(0, 20)  # Omezíme maximální hodnotu y pro čitelnost
ax1.set_xlabel('x')
ax1.set_ylabel('f(x)')
ax1.set_title('Lineární měřítko')
ax1.legend()
ax1.grid(True)
ax1.axvline(x=1, color='r', linestyle='--', alpha=0.3, label='x=1 (asymptota)')

# Graf 2: Logaritmické měřítko pro lepší zobrazení chování blízko x=1
for i, n in enumerate(n_values):
    y = f(x, n)
    # Vyhneme se hodnotám, kde je y přesně nulové (pro x=0)
    valid_indices = y > 1e-10
    ax2.semilogy(x[valid_indices], y[valid_indices], label=f'n = {n}', color=col

ax2.set_xlim(0, 1)
ax2.set_xlabel('x')
ax2.set_ylabel('f(x) - logaritmické měřítko')
ax2.set_title('Logaritmické měřítko')
ax2.legend()
ax2.grid(True)
ax2.axvline(x=1, color='r', linestyle='--', alpha=0.3)

# Hlavní nadpis
plt.suptitle('Funkce $f(x) = \\frac{x^{n+1}}{1-x}$ na intervalu $[0, 1)$', fonts
plt.tight_layout()
plt.show()

# Detailní pohled na chování funkce poblíž nuly
plt.figure(figsize=(10, 6))
```

```python
x_detail = np.linspace(0, 0.5, 1000)  # Detailní pohled na interval [0, 0.5]

for i, n in enumerate(n_values):
    y = f(x_detail, n)
    plt.plot(x_detail, y, label=f'n = {n}', color=colors[i])

plt.xlim(0, 0.5)
plt.ylim(0, 0.5)  # Omezíme maximální hodnotu y pro lepší detail
plt.xlabel('x')
plt.ylabel('f(x)')
plt.title('Detail chování funkce $f(x) = \\frac{x^{n+1}}{1-x}$ poblíž x=0')
plt.legend()
plt.grid(True)
plt.show()
```




## Weierstrassovo kritérium

**Sentence.** The function series $\sum_n f_n(x)$ is uniformly convergent on the interval $I$ if there exists a so-called **majorant convergent number series** $\sum_{n=1}^{\infty} a_n$ satisfying the inequality $I$ on the interval $I$ for every $n \in \mathbb{N}$

$$|f_n(x)| \leq a_n.$$

## Examples

Let us examine the uniform convergence of the following series:

a) $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$.

b) $\sum_{n=1}^{\infty} \frac{\cos nx}{e^{nx}}$.

c) $\sum_{n=1}^{\infty} \arctan \frac{2x}{x^2+n^3}$.

### Solution a)

The members of the function series are the functions $f_n(x) = \frac{\sin nx}{n^2}$, which are defined on the interval $I = (-\infty, \infty)$. Furthermore:

$$|\frac{\sin nx}{n^2}| \leq \frac{1}{n^2},$$

for each $x \in I$ and for each $n \in \mathbb{N}$. Let us now use the W. criterion with convergent majorant $\sum_n \frac{1}{n^2}$. According to the W. criterion, the series converges uniformly on the interval $I$.

### Solution b)

b) $\sum_{n=1}^{\infty} \frac{\cos nx}{e^{nx}}$.

Choose any $a > 0$ and prove that for every $n \in \mathbb{N}$ and for every $x \in (a, \infty)$,

$$\left|\frac{\cos nx}{e^{nx}}\right| \leq \frac{1}{e^{an}}.$$

$$\left|\frac{\cos nx}{e^{nx}}\right| \leq \frac{1}{e^{nx}} \leq \frac{1}{e^{na}}, \quad x > a.$$

Now we will investigate the convergence of the majorant series $\sum_{n=1}^{\infty} \frac{1}{e^{na}}$. We will use the quotient limit criterion:

$$L = \lim_n |\frac{a_{n+1}}{a_n}| = \lim_n |\frac{1}{e^{(n+1)a}} / \frac{1}{e^{na}}| = \frac{1}{e^a}.$$

Apparently from here $L < 1$. The majorant series is therefore a convergent series and the W. criterion implies uniform convergence of the given series on every interval $(a, +\infty)$, where $a > 0$.

### Solution c)

c) $\sum_{n=1}^{\infty} \arctan \frac{2x}{x^2+n^3}$.

The functions $\arctan \frac{2x}{x^2+n^3}$ are apparently odd smooth functions on the interval $I = (-\infty, \infty)$. Furthermore, these functions are bounded on the interval $I$. Furthermore, for each $n \in \mathbb{N}$ is

$$\lim_{x \to \pm\infty} f_n(x) = 0.$$

Thus, for each $n$ it is possible to determine the absolute maximum/minimum of the $f_n(x)$ function on the $I$. interval, we determine the zero points of the first derivative for this purpose:

$$f_n'(x) = \frac{2n^3 - 2x^2}{(x^2+n^3)^2 + 4x^2}.$$

Setting the first derivative equal to zero leads to the equation:

$$2n^3 - 2x^2 = 0.$$

The zero points of the first derivative are the points: $x = \pm n\sqrt{n}$. The local maximum occurs for $x = n\sqrt{n}$. So the maximum is equal to:

$$\phi(n) = f_n(n\sqrt{n}) = \arctan(\frac{2n\sqrt{n}}{(n\sqrt{n})^2 + n^3}) = \arctan(\frac{\sqrt{n}}{n^2}).$$

Thus, for each $n \in \mathbb{N}, x \in \mathbb{R}$:

$$\left| \arctan \frac{2x}{x^2+n^3} \right| \leq \arctan(\frac{\sqrt{n}}{n^2}) \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}.$$

The last inequality follows from the inequality:

$$|\arctan x| \leq |x|, \quad x \in \mathbb{R}.$$

A majorant series can therefore be a convergent series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. The convergence of the last series can be verified using the integral criterion.

# Power series

The most important example of a functional series is the so-called power series of the form:

$$\sum_{n=0}^{\infty} c_n \cdot (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots \qquad (*)$$

We determine the **domain of convergence** of power series in general as for functional series. The so-called **radius of convergence** is used.

The radius of convergence can be determined using the formula:

$$R = \lim_{n} \left| \frac{c_n}{c_{n+1}} \right|.$$

Or

$$R=\lim_{n}\frac{1}{\sqrt[n]{|c_n|}},$$

if said limits exist. If both limits exist, then these limits are equal.

If $R$ is the radius of convergence of the power series (*), then an open interval

$$(a-R,a+R)$$

we call the **convergence interval.**

# Examples

We determine the domain of convergence of power series:

a) $\sum_{n=0}^{\infty}\frac{(x+1)^n}{n\cdot 4^{n-1}}$.

b) $\sum_{n=1}^{\infty}\frac{10^n\cdot x^n}{\sqrt{n}}$.

c) $\sum_{n=0}^{\infty}\frac{3^n}{n+1}(x-4)^n$

d) $\sum_{n=0}^{\infty}\frac{x^n}{(2n)!}$.

## Solution a)

a) $\sum_{n=0}^{\infty}\frac{(x+1)^n}{n\cdot 4^{n-1}}$.

$$c_n=\frac{1}{n\cdot 4^{n-1}},\quad n\in\mathbb{N}.$$

(i) (Radius of Convergence):

$$R=\lim_{n}\left|\frac{c_n}{c_{n+1}}\right|=\lim_{n}\left[\frac{1}{n\cdot 4^{n-1}}:\frac{1}{(n+1)\cdot 4^n}\right]=\lim_{n}\frac{4^n\cdot(n+1)}{4^{n-1}\cdot n}=4.$$

The convergence interval is the $I=(a-R,a+R)=(-1-4,-1+4)=(-5,3)$. interval

We now examine the convergence at the endpoints of the convergence interval.

$x=-5$

$$\sum_{n=0}^{\infty}\frac{(x+1)^n}{n\cdot 4^{n-1}}|_{x=-5}=\sum_{n=0}^{\infty}\frac{(-5+1)^n}{n\cdot 4^{n-1}}=$$
$$=\sum_{n=0}^{\infty}\frac{(-4)^n}{n\cdot 4^{n-1}}=\sum_{n=0}^{\infty}4\cdot(-1)^n\cdot\frac{1}{n}.$$

The last series is an alternating convergent series.

$x = 3$

$$\sum_{n=0}^{\infty} \frac{(x+1)^n}{n \cdot 4^{n-1}} |_{x=3} = \sum_{n=0}^{\infty} \frac{4}{n}.$$

This series is a divergent series. The domain of convergence of the initial series is therefore the interval: $\langle -5, 3)$.

## Solution b)

b) $\sum_{n=1}^{\infty} \frac{10^n \cdot x^n}{\sqrt{n}}$.

Here $c_n = \frac{10^n}{\sqrt{n}}$. Then

$$R = \lim_n |\frac{c_n}{c_{n+1}}| = \lim_n \left[ \frac{10^n}{\sqrt{n}} : \frac{10 \cdot 10^n}{\sqrt{n+1}} \right] = \lim_n \frac{1}{10} \sqrt{\frac{n+1}{n}} = 1/10.$$

The interval of convergence is $(-\frac{1}{10}, \frac{1}{10})$. Furthermore, the series converges for $x = -\frac{1}{10}$ and diverges for $x = \frac{1}{10}$.

In [2]:
```python
import scipy.integrate as integrate
import numpy as np
import matplotlib.pyplot as plt

# Definice funkce
def integrand(x):
    return np.exp(-x**2)

# Výpočet integrálu
result, error = integrate.quad(integrand, 1, 10)
result, error
```

Out[2]:
```
(0.13940279264033098, 3.928467470000696e-15)
```

## Solution c)

c) $\sum_{n=0}^{\infty} \frac{3^n}{n+1}(x-4)^n$

Let's use the ratio criterion again:

$$R = \lim_n |\frac{c_n}{c_{n+1}}| =$$
$$\lim_n \left[ \frac{3^n}{n+1} : \frac{3 \cdot 3^n}{n+2} \right] = 1/3.$$

Now we determine the endpoints of the convergence interval:

$$a - R = 4 - 1/3 = 11/3, \quad a + R = 4 + 1/3 = 13/3.$$

The convergence interval is the interval

$$(11/3, 13/3).$$

We now examine the convergence at the endpoints of the conv. interval.

i) $x = 11/3$ : by substituting we get a number series:

$$\sum_{n=0}^{\infty} \frac{3^n}{n+1}(11/3 - 4)^n = \sum_{n=0}^{\infty} \frac{3^n}{n+1}(\frac{11-12}{3})^n = \sum_{n=0}^{\infty} \frac{3^n}{n+1}(-1/3)^n = \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot (-1)^n.$$

This series is convergent as follows from the Leibniz criterion.

ii) $x = 13/3$ :

$$\sum_{n=0}^{\infty} \frac{3^n}{n+1}(x-4)^n|_{x=13/3} = \sum_{n=0}^{\infty} \frac{3^n}{n+1}(13/3 - 4)^n = \sum_{n=0}^{\infty} \frac{3^n}{n+1}(1/3)^n = \sum_{n=0}^{\infty} \frac{1}{n+1} = +\infty.$$

So the domain of convergence is a semi-closed interval $\langle 11/3, 13/3)$.

## Solution d)

d) $\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$. Zde

$$c_n = \frac{1}{(2n)!}.$$

Now let's calculate the radius of convergence:

$$R = \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \left[\frac{1}{(2n)!} : \frac{1}{(2(n+1))!}\right]$$
$$= \lim_{n\to\infty} \left[\frac{1}{(2n)!} : \frac{1}{(2n+2)!}\right] = \lim_{n\to\infty} \frac{(2n+2)(2n+1)(2n)!}{(2n)!} = \lim_{n\to\infty} (2n+2)(2n+1$$

The convergence interval is therefore the interval $(-\infty, +\infty)$. It also follows from this that the domain of convergence is the interval $(-\infty, +\infty)$.

# Properties of power series

Let the power series $\sum_n c_n \cdot x^n$ have radius of convergence $R > 0$. Then the following holds:

i) The sum $S(x) = \sum_n c_n \cdot x^n$ is a continuous function on the convergence interval.

ii) The power series converges uniformly to the function $S(x)$ on every closed interval $\langle a, b\rangle \subset (-R, R)$.

iii) A power series can be derived or integrated term by term any number of times in each closed interval $\langle a, b\rangle \subset (-R, R)$. The series formed by differentiating or integrating a given power series have a common radius of convergence with it.

iv) If $\sigma(x) = \sum_{n=0}^{\infty}(c_n x^n)'$ is the sum of the series formed by differentiating a power series term by term, then:

$$\sigma(x) = S'(x).$$

If $\sigma(x) = \sum_{n=0}^{\infty} \int c_n x^n dx$ is the sum of a series formed by integrating a power series term by term, then:

$$\sigma(x) = \int S(x)dx + C,$$

where $C$ is determined by substituting $x = 0$ into the last equality.

## Examples

For the following power series, we determine the domain of convergence and the sum of the series:

a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

b) $\sum_{n=1}^{\infty} n \cdot x^{n-1}$.

### Solution a)

a) $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

i) (radius of convergence)

$$R = \lim_n \left|\frac{c_n}{c_{n+1}}\right| = \lim_n \left[\frac{1}{n} / \frac{1}{n+1}\right] = 1.$$

So the convergence interval is the interval: $(-1, 1)$.

ii) (Convergence at endpoints of conv. interval):

$$x = -1:$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n}|_{x=-1} = \sum_{n=1}^{\infty} \frac{1}{n} \cdot (-1)^n.$$

This series converges due to the Leibniz criterion.

$$x = 1$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n}|_{x=1} = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

So the domain of convergence is equal to the interval $\langle -1, 1)$.

iii) (Sum of series): The given series was formed by integrating the series

$$\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}.$$

If we now mark $\sigma(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$, then:

$$\sigma(x) = \sum_{n=1}^{\infty} \int x^{n-1} dx = \int \frac{1}{1-x} dx = -\ln(1-x) + C.$$

We determine the constant $C$ by substituting $x = 0$ into the previous relation:

$$0 = \sigma(0) = -\ln(1-0) + C \implies C = 0.$$

### Solution b)

b) $\sum_{n=1}^{\infty} n \cdot x^{n-1}$.

i) (Radius of Convergence):

$$R = \lim_n \left| \frac{c_n}{c_{n+1}} \right| = \lim_n \frac{n}{n+1} = 1.$$

ii) (Convergence at endpoints of conv. interval):

$$x = -1$$

$$\sum_{n=1}^{\infty} n \cdot x^{n-1}|_{x=-1} = \sum_{n=1}^{\infty} n \cdot (-1)^{n-1} \implies$$

the series diverges.

$$x = 1$$

$$\sum_{n=1}^{\infty} n \cdot x^{n-1}|_{x=1} = \sum_{n=1}^{\infty} n \implies$$

the series diverges. So the domain of convergence is equal to the open interval

$$(-1, 1).$$

iii) (Sum of series): The initial series was created by deriving a power series (it is a geometric series with a quotient of $q = x$ and where the first term of the series is equal to $x$.) Thus:

$$S(x) = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x} \implies$$

$$\sigma(x) = \sum_{n=1}^{\infty} n \cdot x^{n-1} = \sum_{n=1}^{\infty} (x^n)' = \left(\frac{x}{1-x}\right)' = \frac{1}{(1-x)^2}.$$

# Taylor and MacLaurin series

Assume that the function $f : (a-\delta, a+\delta) \to \mathbb{R}$ has derivatives of all orders at the point $a$. Then a power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n$$

we call the **Taylor expansion of the function $f$ at the point** $x = a$. In the case where $a = 0$, is, we are talking about the so-called **MacLauriv development of the $f$ function.**

The Taylor expansion of $f$ at $x = a$ is a power series centered at $x = a$.

## The rest in Taylor's development

We call the function $R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{n!} \cdot (x-a)^k$ **the remainder in the Taylor expansion.** Here $n \in \mathbb{N},\ x \in (a-\delta, a+\delta)$.

A necessary and sufficient condition for the Taylor series to converge and sum to $f(x)$ is

$$\lim_{n\to\infty} R_n(x) = 0.$$

The rest of $R_n(x)$ can be expressed in the so-called Lagrangian form for T. development at the point $x = a$ as follows

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot (x-a)^{n+1},\ \text{kde}\ \xi = a + \theta \cdot (x-a),\ \theta \in (0,1).$$

If the function $f$ has derivatives of all orders at the point $a$ and if there exists a $\delta > 0$ such that for every $x \in (a-\delta, a+\delta)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n,$$

then we say that the **function $f$ is analytic at the point** $x = a$**.**

**Theorem (A sufficient condition for analyticity).** Let the function $f$ at the point $a$ have derivatives of all orders, and let the condition for $K > 0$ be satisfied on a certain neighborhood of the point $a$ for all $n \in \mathbb{N}$

$$|f^{(n)}(x)| \leq K.$$

Then the function $f$ is analytic at the point $x = a$.

### konec 23.4.24

## Examples a)

a) Find the MacLaurin expansion of the function $\cos x$ and show that the function $\cos x$ is an analytic function at the point $x = 0$.

### Solution a)

Let's calculate successively the ferivations at the point $x = 0$.

- $f(0) = \cos(0) = 1,$
- $f'(0) = \cos'(0) = -\sin(0) = 0,$
- $f''(0) = -\sin'(0) = -\cos(0) = -1,$
- $f'''(0) = \sin(0) = 0,$
- $f^{(4)}(0) = \sin'(0) = \cos(0) = 1,$
- ...

Substituting into the general form of the MacLaurin expansion, we get a power series:

$$1 + \frac{0}{1!} \cdot x + \frac{-1}{2!} x^2 + \frac{0}{3!} \cdot x^3 + \frac{1}{4!} \cdot x^4 + \cdots + \frac{(-1)^n}{(2n)!} \cdot x^{2n} + \cdots$$

Furthermore, for each $x \in \mathbb{R}$:

$$|f^{(n)}(x)| \leq 1.$$

Thus, the function $f(x) = \cos(x)$ is analytic at the point $x = 0$. So you can write around zero:

$$\cos(x) = 1 + \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{(2n)!} \cdot x^{2n}.$$

## Examples b)

a) Now show separately that

$$\sin x = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{(2n-1)!} \cdot x^{2n-1}.$$

### Solution b)

The MacLaurin expansion of $\sin x$ can be advantageously obtained by integrating the expansion of $\cos x$.

$$\cos x = 1 + \frac{0}{1!} \cdot x + \frac{-1}{2!} x^2 + \frac{0}{3!} \cdot x^3 + \frac{1}{4!} \cdot x^4 + \cdots + \frac{(-1)^n}{(2n)!} \cdot x^{2n} + \cdots$$

$$\sin x = x - \frac{1}{2!} \cdot \frac{x^3}{3} + \frac{1}{4!} \cdot \frac{x^5}{5} - \frac{1}{6!} \cdot \frac{x^7}{7} + \cdots$$

$$\sin x = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{(2n-1)!} \cdot x^{2n-1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

## Examples c)

Let's verify the following expansion of the function $y = e^x$.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots, \quad x \in \mathbb{R}. \qquad (*)$$

### Solution c)

- $f(0) = e^0 = 1,$
- $f'(0) = e^0 = 1,$
- ...
- $f^{(n)}(0) = e^0 = 1.$

Hence the shape of MacLaurin's development:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

This is a power series whose radius of convergence is $R = \infty$. Furthermore, let $\delta > 0$ be arbitrarily fixed. Then for every $x \in (-\delta, \delta)$ :

$$|f^{(n)}(x)| = e^x \leq e^\delta = K.$$

This proves that the exponential is an analytic function at $x = 0$ and the above formula (*) holds.

In [ ]:
```
######### Zde je hlavička ################
from math import *
import sympy as sym
from sympy.plotting import plot3d
from IPython.display import Math, display, Latex
import numpy as np
# sym.init_printing()
##################################################
```

In [ ]:
```
from sympy.abc import x
sym.sqrt(1 / 1-x**2).series(n=15)
```

Out[ ]: $1.0 - 0.5x^2 - 0.125x^4 - 0.0625x^6 - 0.0390625x^8 - 0.02734375x^{10} - 0.0205078125x^{12}$

**Theorem.** Assume that the function $f$ is equal to the sum of the power series at the point $x = a$, i.e. that for $R > 0$ is

$$f(x) = \sum_{n=0}^{\infty} c_n \cdot (x-a)^n, \quad \text{pro každé } x \in (a-R, a+R).$$

Then for the $c_n$ coefficients:

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}.}$$

We now summarize Tylor's developments of some important elementary functions.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots, \; R = 1,$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \; R = \infty,$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, \; R = \infty,$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, \; R = \infty,$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \; R = 1,$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} \cdot x^n = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \ldots, \; R =$$

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots, \; R = 1.$$

# Examples

**Example (a).** Let's find the Maclaurian expansion of the function $f(x) = x \cos x$.

## Solution a)

$$x \cos x = x \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n)!}, \forall x \in \mathbb{R}.$$

**Example (b).** Let's find the Maclaurian expansion of the function $f(x) = \ln(1+3x^2)$.

## Solution b)

$$\ln(1+3x^2) = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{(3x^2)^n}{n} = 3x^2 - \frac{9x^4}{2} + \frac{27x^6}{3} - \frac{81x^8}{4} + \ldots.$$

From the previous table, we know that the Maclaurian expansion of the $\ln(1+x)$ function is equal to

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots.$$

Here the radius of convergence is equal to $R = 1$. So in our case $|3x^2| < 1$, i.e. $|x| < 1/\sqrt{3}$. must apply

**Example (c).** Determine the function whose Maclaurin expansion is equal to

$$\sum_{n=0}^{\infty}(-1)^n \frac{2^n x^n}{n!}.$$

## Solution c)

Let's modify the given power series slightly:

$$\sum_{n=0}^{\infty}(-1)^n \frac{2^n x^n}{n!}=\sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}=e^{-2x}.$$

**Example (d).** Let's find the sum of the series:

$$\frac{1}{1 \cdot 2}-\frac{1}{2 \cdot 2^2}+\frac{1}{3 \cdot 2^3}-\frac{1}{4 \cdot 2^4}+\ldots.$$

## Solution d)

Using the summation symbol, the given series can be written as

$$\sum_{n=1}^{\infty}(-1)^{n-1} \frac{1}{n \cdot 2^n}.$$

This series can now be rewritten into the form

$$\sum_{n=1}^{\infty}(-1)^{n-1} \frac{1}{n \cdot 2^n}=\sum_{n=1}^{\infty}(-1)^{n-1} \frac{\left(\frac{1}{2}\right)^n}{n}.$$

From the previous table of expansions, it follows that the Maclaurian expansion of the $\ln(1+x)$ function is equal to

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots.$$

So it applies to $x=1/2$

$$\ln(1+1/2)=\ln(3/2)=\frac{1}{2}-\frac{1}{2^2 \cdot 2}+\frac{1}{3 \cdot 2^3}-\frac{1}{4 \cdot 2^4}+\ldots.$$

**Example (e).** Let's find the indefinite integral

$$\int e^{-x^2} dx.$$

## Solution e)

Let's first find the Maclaurin expansion of the function $e^{-x^2}$. Let's start from the Maclaurin expansion of the function $e^x$ :

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

Now let's substitute $-x^2$ after $x$ in the development above:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots.$$

Now we can integrate term by term:

$$\int e^{-x^2} dx = \int \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\right) dx = C + x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \ldots.$$

I mean

$$\int e^{-x^2} dx = C + x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \ldots. = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!}.$$

This power series converges for all $x \in \mathbb{R}$ because the radius of convergence is $R = \infty$.
$\square$

**Example (f).** Calculate approximately the value of a definite integral

$$\int_0^1 e^{-x^2} dx.$$

## Solution f)

The previous example gave us Maclaur's expansion of the primitive function to the function $e^{-x^2}$. Thus

$$\int_0^1 e^{-x^2} dx = \left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \ldots\right]_0^1 = 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \ldots.$$

Thus, if we use the first 5 terms of the development, we will receive approximately the value of a definite integral

$$\int_0^1 e^{-x^2} dx \approx 1 - \frac{1}{3} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \frac{1}{9 \cdot 4!} \approx 0.7475.$$

Since the series is an alternating convergent series, we can estimate the approximation error. It applies to the approximation error

$$|R| < \frac{1}{11 \cdot 5!} < 0.001.$$

**Example (g).** Using the Maclaurin expansion of the $e^x$ function, determine the limit:

$$\lim_{x\to 0} \frac{e^x - 1}{x}.$$

## Solution g)

We know that the Maclaurian expansion of the function $e^x$ is equal to

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

From here we get an expression for each $x \in \mathbb{R},\ x \neq 0:$

$$\begin{aligned}\frac{e^x - 1}{x} &= \left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) - 1\right] \cdot \frac{1}{x} \\ &= \left[x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right] \cdot \frac{1}{x} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots = 1 + \frac{x}{2} + \frac{x^2}{6} + \dots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}.\end{aligned}$$

The radius of convergence of this series is $R = \infty$. Thus the function $S(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$ is defined on $\mathbb{R}$ and is a continuous function. So we can write

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} S(x) = S(0) = 1.$$

**Example (h).** Using the Maclaurin expansion of the $e^x$ function, determine the limit:

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}.$$

## Solution h)

We know that the Maclaurian expansion of the function $e^x$ is equal to

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

From here we get an expression for each $x \in \mathbb{R},\ x \neq 0:$

$$\begin{aligned}\frac{e^x - 1 - x}{x^2} &= \left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) - 1 - x\right] \cdot \frac{1}{x^2} \\ &= \frac{1}{x^2}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) = \frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \dots\end{aligned}$$

The radius of convergence of this series is $R = \infty$. Thus the function $S(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n+2)!}$ is defined on $\mathbb{R}$ and is a continuous function. So we can write

$$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} S(x) = S(0) = \frac{1}{2!} = \frac{1}{2}.$$

**Example (i).** Using the Maclaurin expansion of the $\ln(1+x)$ function, determine the limit:

$$\lim_{x \to 0} \frac{1-x}{x^2}(x - \ln(1+x)).$$

## Solution i)

We know that the Maclaurian expansion of the function $\ln(1+x)$ is equal to

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots.$$

From here we get an expression for each $x \in (-1, 1),\ x \neq 0:$

$$\begin{aligned}
\frac{1-x}{x^2}(x - \ln(1+x)) &= \frac{1-x}{x^2}\left(x - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right)\right) \\
&= \frac{1-x}{x^2}\left(\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5} + \dots\right) \\
&= (1-x)\left(\frac{1}{2} - \frac{x}{3} + \frac{x^2}{4} - \frac{x^3}{5} + \dots\right) \\
&= (1-x)\sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n+2}.
\end{aligned}$$

Now let's put $S(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n+2}$. The radius of convergence of this series is $R = 1$. So the function $S(x)$ is defined on the interval $(-1, 1)$ and is a continuous function. So we can write

$$\begin{aligned}
\lim_{x \to 0} \frac{1-x}{x^2}(x - \ln(1+x)) &= \lim_{x \to 0}(1-x)S(x) \\
&= \lim_{x \to 0}(1-x)\lim_{x \to 0} S(x) \\
&= (1-0)S(0) = 1 \cdot \frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

**Example (j).** Using the previous limits, determine the limit of the sequence:

$$\lim_{n \to \infty} n\left(1 - e\left(\frac{n+1}{n+2}\right)^{n+1}\right).$$

## Solution j)

First, let's modify the given expression:

$$
\begin{aligned}
n\left(1-e\left(\frac{n+1}{n+2}\right)^{n+1}\right) &= n\left(1-e\cdot e^{\ln\left(\frac{n+1}{n+2}\right)^{n+1}}\right)\\
&= n\left(1-e\cdot e^{\ln\frac{1}{\left(\frac{n+2}{n+1}\right)^{n+1}}}\right)\\
&= n\left(1-e\cdot e^{\ln\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}}\right)\\
&= -n\left(e^{1+\ln\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}}-1\right)\\
&= -n\left(1+\ln\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}\right)\cdot\frac{e^{1+\ln\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}}-1}{1+\ln\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}}\\
&= -n\left(1-(n+1)\ln\left(1+\frac{1}{n+1}\right)\right)\cdot\frac{e^{1+\ln\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}}-1}{1+\ln\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}}\\
&= -n(n+1)\left(\frac{1}{n+1}-\ln\left(1+\frac{1}{n+1}\right)\right)\cdot\frac{e^{1+\ln\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}}-1}{1+\ln\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}}
\end{aligned}
$$

In addition, the following applies:

$$\lim_{n\to\infty}\left(1+\ln\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}\right)=1+\ln\frac{1}{e}=0.$$

From there it follows that

$$\lim_{n\to\infty}\frac{e^{1+\ln\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}}-1}{1+\ln\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}}=\lim_{x\to 0}\frac{e^x-1}{x}=1.$$

Now consider the function $f(x)=\frac{1-x}{x^2}(x-\ln(1+x))$. Then for every $n\in\mathbb{N}$ holds

$$f(1/(n+1))=n(n+1)\left(\frac{1}{n+1}-\ln\left(1+\frac{1}{n+1}\right)\right).$$

Then:

$$\frac{1}{2}=\lim_{x\to 0}f(x)=\lim_{n\to\infty}f\left(\frac{1}{n+1}\right)=\lim_{n\to\infty}n(n+1)\left(\frac{1}{n+1}-\ln\left(1+\frac{1}{n+1}\right)\right).$$

Finally, from the above calculations it follows that

$$\lim_{n\to\infty} n\left(1 - e\left(\frac{n+1}{n+2}\right)^{n+1}\right) = \lim_{n\to\infty} -n(n+1)\left(\frac{1}{n+1} - \ln\left(1+\frac{1}{n+1}\right)\right)\cdot\frac{e^{1+\ln \cdots}}{1+\ln \cdots}$$

$$= -\frac{1}{2}\cdot 1 = -\frac{1}{2}.$$

□