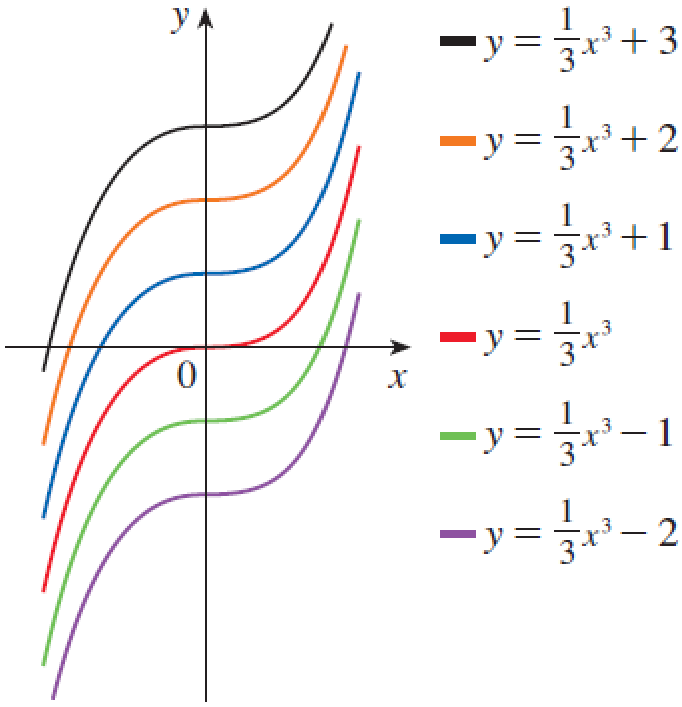
Primitive function

Definition. A function F is called a **primitive function to a function f on an interval I,** if $F'(x) = f(x)$. holds for every $x \in I$

Example. Consider the function $f: (-\infty, \infty) \to \mathrm{R}$ given by the prescription $f(x) = x^2$. Then the function $F(x) = \frac{1}{3}x^3$. Then for every $x \in (-\infty, \infty)$ $F'(x) = x^2 = f(x)$. Next, consider the function $G(x) = \frac{1}{3}x^3 + 100$. The function G is also a primitive function to the function f because for every $x \in (-\infty, \infty)$. More generally, if $C \in \mathrm{R}$ is an arbitrary constant and H is function defined by $H(x) = \frac{1}{3}x^3 + C$, then H is a primitive function to f.



Theorem. If F_0 is a primitive function to the function f on the interval I, then the general form of the primitive function is:

$$F(x) = F_0(x) + C,$$

where C is an arbitrary constant.

This theorem is a consequence of the following theorem:

Theorem. Suppose that for each $x \in I$:

$$F'(x) = G'(x).$$

Then it holds on the interval I that the difference $F - G$ is a constant function.

Proof. This is a consequence of the so-called Lagrange theorem about the mean value. Apparently for every $x \in I$:

$$(F(x) - G(x))' = F'(x) - G'(x) = 0.$$

It follows that there exists a constant $C \in \mathbb{R}$ such that for every $x \in I$ there is a $F(x) - G(x) = C$. $\square$

Example. We have already calculated the derivative

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}, \quad \forall x \neq 0.$$

By the previous theorem, a general primitive function is a function

$$F(x) = \ln|x| + C$$

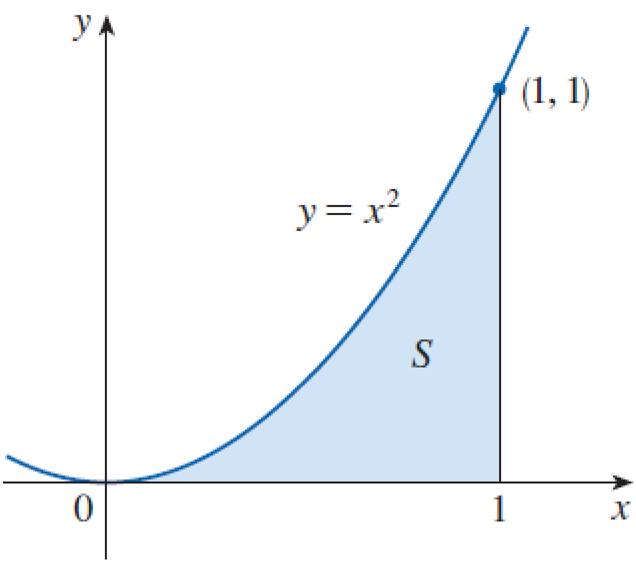
on any interval I not containing the point $x = 0$. Especially on the intervals $(-\infty, 0)$ and $(0, \infty)$.

Integrals

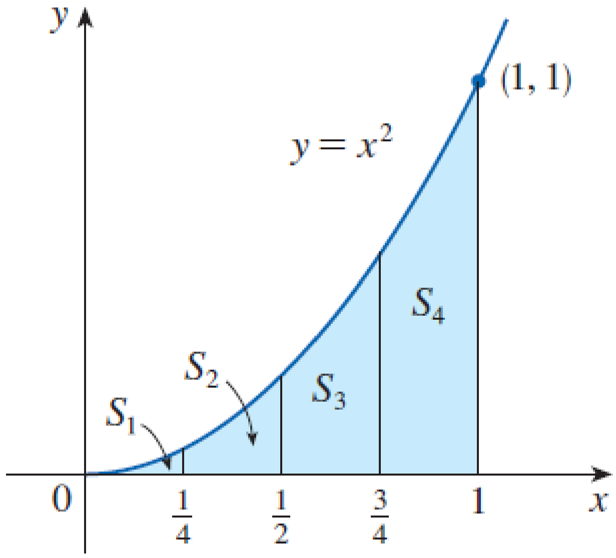
Content calculation problem

Note. The first serious example of a measure is the so-called **Jordan-Pean measure**, named after the mathematicians Camillo Jordan and Guiseppe Pean. Later, the French mathematician Henri Lebesgue came up with a more general concept of measure, which we now call **Lebesgue's measure.**

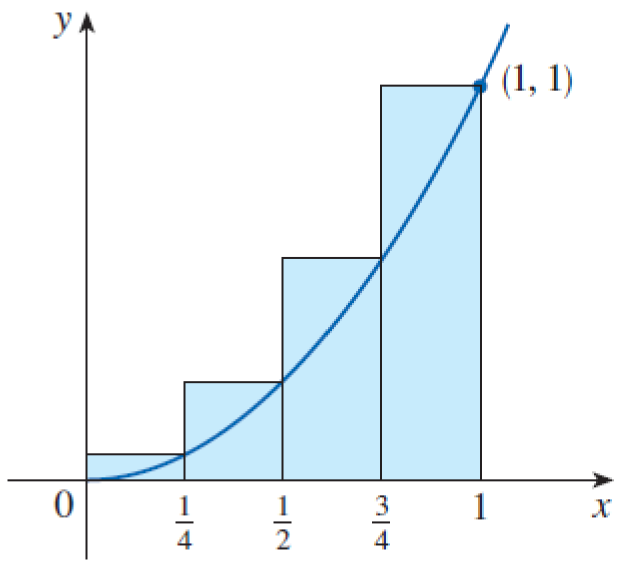
Consider the plane figure S shown in the figure below.



Now let's divide the figure into parts using vertical lines $x = \frac{1}{4}, x = \frac{1}{2}, x = \frac{3}{4}$. See figure below.



Next, let's replace each curvilinear trapezoid S_1, S_2, S_3 and S_4 with a rectangle as shown in the following image.



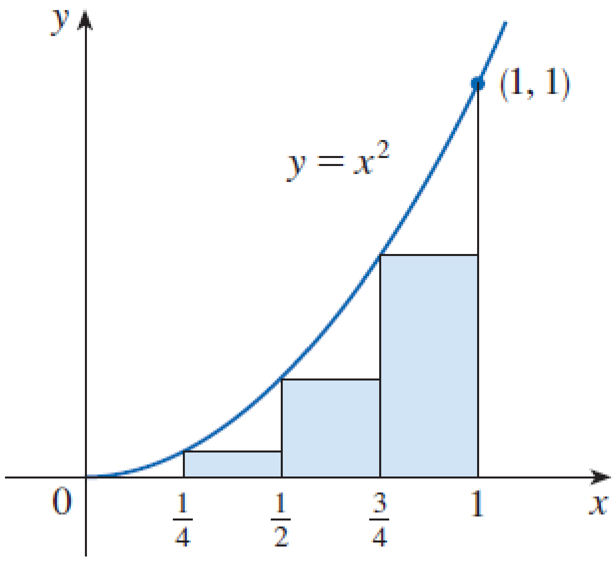
I divided the interval into four intervals of equal length: $\langle 0, \frac{1}{4}\rangle$, $\langle \frac{1}{4}, \frac{1}{2}\rangle$, $\langle \frac{1}{2}, \frac{3}{4}\rangle$ and $\langle \frac{3}{4}, 1\rangle$. The height of each rectangle that approximates the figure S_i is equal to the functional value of the function $f(x)$ at the right endpoint of the corresponding subinterval forming the base of the rectangle. Thus, each of these rectangles has a width of $\frac{1}{4}$ and the heights are successively equal to $\left(\frac{1}{4}\right)$, $\left(\frac{1}{2}\right)$, $\left(\frac{3}{4}\right)$ and 1^2. Let R_4 denote the sum of the contents of the individual rectangles, i.e.

$$R_4 = \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 + \frac{1}{4} \cdot (1)^2 = \frac{15}{32}.$$

From the image above, it is clear that the content of A of the S shape is smaller than R_4, so

$$A < \frac{15}{32}.$$

Now let's approximate the shape S using rectangles whose height will this time be equal to the functional value of the function $f(x) = x^2$ at the left endpoint of each subinterval $\langle 0, \frac{1}{4} \rangle$, $\langle \frac{1}{4}, \frac{1}{2} \rangle$, $\langle \frac{1}{2}, \frac{3}{4} \rangle$ and $\langle \frac{3}{4}, 1 \rangle$. See the following figure:



The sum of the contents of the individual approximating rectangles is analogously equal to:

$$L_4 = \frac{1}{4} \cdot 0^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 = \frac{7}{32}.$$

This is how we got a lower estimate of the content of the S shape. Applies to:

$$\frac{7}{32} < A < \frac{15}{32}.$$

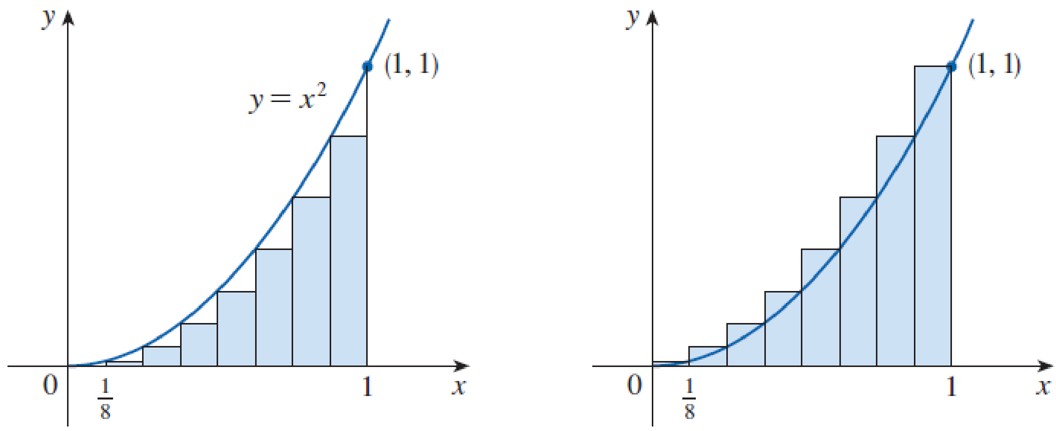
It is of course possible to repeat the calculation for a finer approximation of the shape using more sub-intervals. For example, if we double the number of approximating rectangles, we get estimates of:

$$0.2734375 < A < 0.3984375$$

We get even more accurate estimates by gradually increasing the number of approximating rectangles. See the following table:

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

In the following image you can see the approximation of the shape for the number $n = 8$.



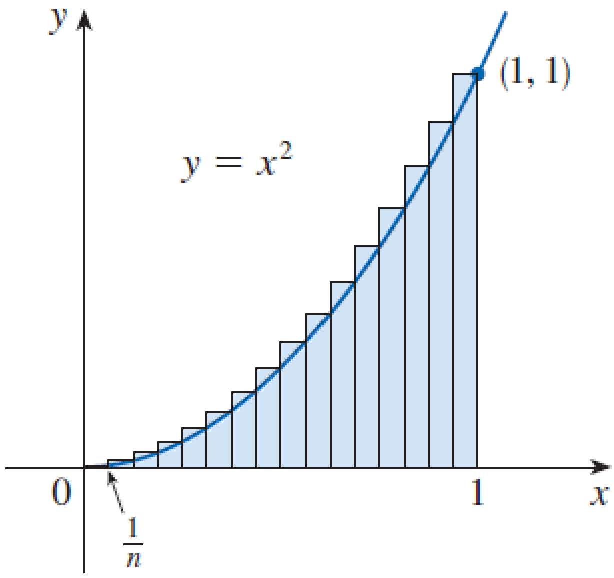
Let us further show that for $n \to \infty$ the sequence (R_n) converges and holds:

$$\lim_{n \to \infty} R_n = \frac{1}{3}.$$

For the calculation, we use the formula for calculating the sum of squares of natural numbers:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

You can try to prove this formula by mathematical induction. R_n will indicate for $n \in \mathrm{N}$ the sum of the contents of the rectangles forming the upper approximation of the figure



$$
\begin{aligned}
R_n &= \frac{1}{n} f\left(\frac{1}{n}\right) + \frac{1}{n} f\left(\frac{2}{n}\right) + \frac{1}{n} f\left(\frac{3}{n}\right) + \cdots + \frac{1}{n} f\left(\frac{n}{n}\right) \\
&= \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2 \\
&= \frac{1}{n} \cdot \frac{1}{n^2}(1^2 + 2^2 + 3^2 + \cdots + n^2) \\
&= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2).
\end{aligned}
$$

Using the summation formula above, we get:

$$R_n = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}.$$

Then, passing to the limit, we get:

$$
\begin{aligned}
\lim_{n \to \infty} R_n &= \lim_{n \to \infty} \frac{(n+1)(2n+1)}{6n^2} \\
&= \lim_{n \to \infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \\
&= \lim_{n \to \infty} \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) \\
&= \frac{1}{6} \cdot 1 \cdot 2 = \frac{1}{3}.
\end{aligned}
$$

In a similar way it can be shown that

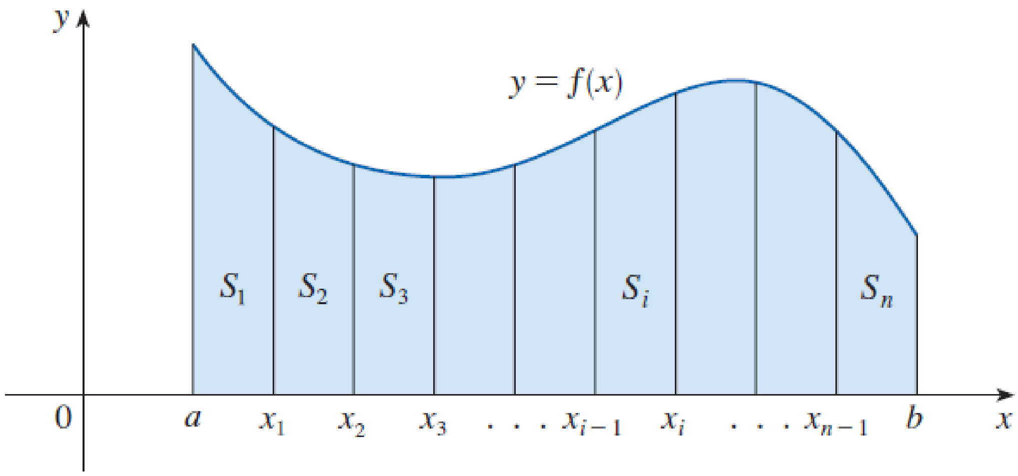
$$\lim_{n \to \infty} L_n = \frac{1}{3}.$$

It now makes sense to define the A content of the S shape as the limit:

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n = \frac{1}{3}.$$

Generalization of the previous example

Consider the non-negative continuous function $f: \langle a, b\rangle \to \mathrm{R}_0^+$ and consider the figure S bounded by the graph of the function f, the axis x and the lines $x = a$ and $x = b$. See figure below



Let's divide the figure into parts $S_1,\ S_2, \cdots, S_n$ the corresponding subintervals $\langle x_{i-1}, x_i\rangle$ have a constant length $\Delta x = \frac{b-a}{n}$. The width of each strip S_i is therefore equal to Δx. The subintervals are the intervals:

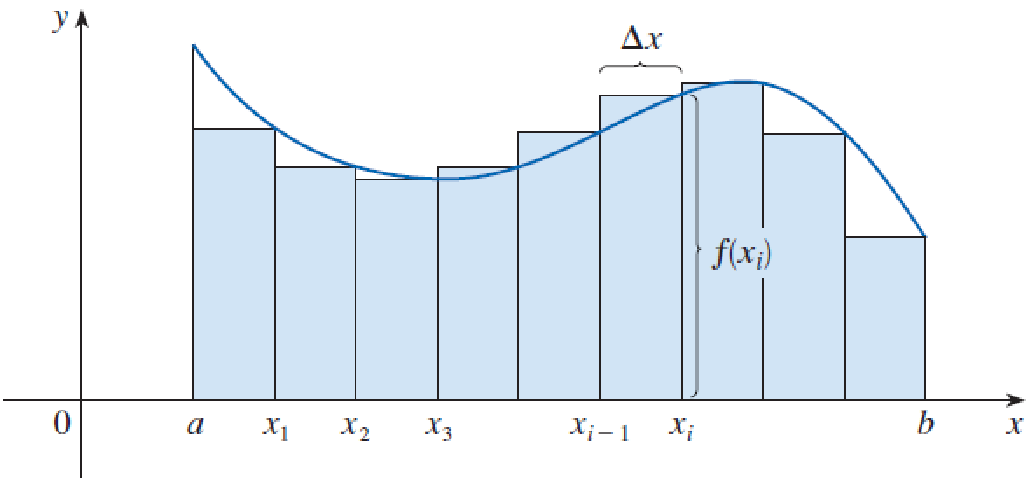
$$\langle x_0, x_1\rangle,\ \langle x_1, x_2\rangle,\ \langle x_2, x_3\rangle, \ldots, \langle x_{n-1}, x_n\rangle,$$

where $x_0 = a$ and $x_n = b$. The right endpoints of the subintervals can be expressed as:

$$x_1 = a + \Delta x, x_2 = a + 2\Delta x, x_3 = a + 3\Delta x,\ \vdots$$

In general $x_i = a + i\Delta x$. Let's approximate the content of each partial shape by the content of a rectangle whose width is equal to Δx and whose height is equal to the function value at the right end point of the respective subinterval $f(x_i.)$ The content $i-$ of that rectangle is therefore equal to the product $f(x_i)\Delta x$. Finally, we will approximate the content of the shape S using the sum R_n contents of individual rectangles:

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$$



Definition. We will define the content A of the shape S bounded by the graph of the continuous function f as the limit:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x].$$

Notes.

1. It is possible to show that due to the assumption of continuity of the function f on the interval $\langle a, b\rangle$ the limit $\lim_{n\to\infty} R_n$ always exists.
2. It can also be shown that if we use the left endpoints of the subintervals, then

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x].$$

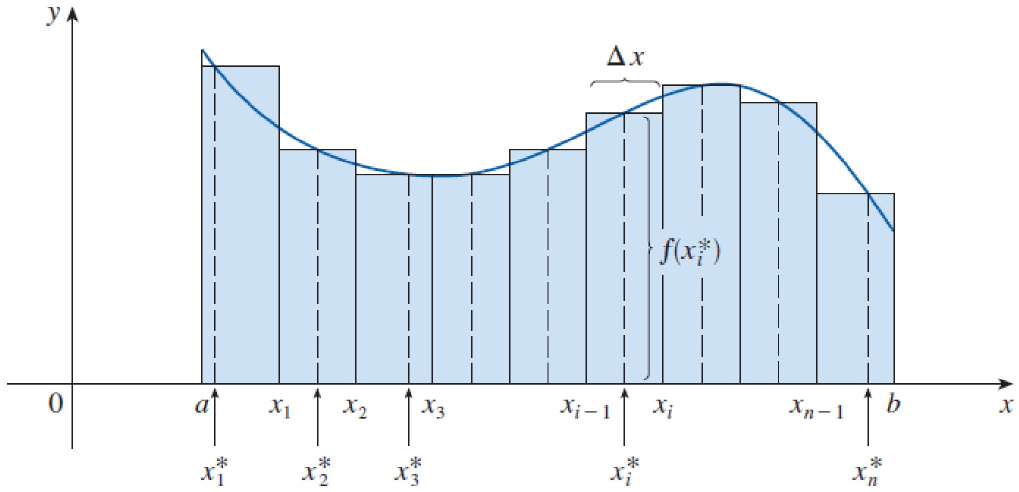
3. Instead of the left or of the right endpoints, we can choose the points $x_1^*, x_2^*, \ldots, x_n^*,$ where for every i there is $x_i^* \in \langle x_{i-1}, x_i\rangle$. Then the content A of the S form can be expressed using more general limits:

$$A = \lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x] = \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*)\Delta x\right].$$

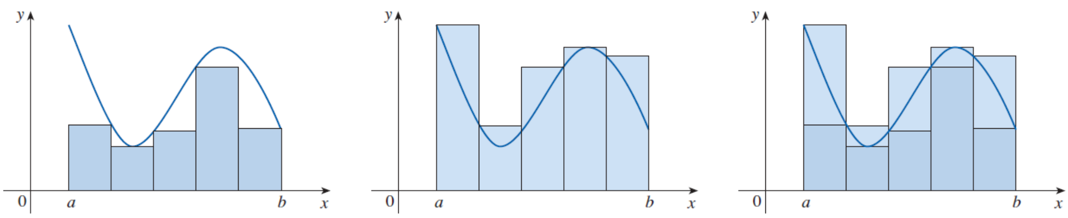
See image below. 4. Due to the continuity of the function f on each of the subintervals, for each i, the selection point $x_i^* \in \langle x_{i-1}, x_i\rangle$ can be chosen so that $f(x_i^*) = \min_{x\in\langle x_{i-1}, x_i\rangle} f(x)$. We thus get the so-called **lower integral sum**, which is a lower estimate of the content of A. Analogously, one can define an **upper integral sum**, which is an upper estimate of the content of the A form S.

Marking. We will mark the content of A with a symbol

$$\int_a^b f(x)dx.$$



Equivalent content definition A. The content of the S form is a uniquely defined value of A, greater than any lower integral sum and less than any upper integral sum.



In [ ]:
```
"""
Vytvořme funkci, která má jako paramtery funkci f, interval [a,b], počet dílčích
intervalů ekvidistantního dělení intervalu [a,b]
a vrací aproximaci integrálu funkce f na intervalu [a,b] pomocí Riemannova integ
"""

import sympy as sp

x = sp.Symbol('x')
import numpy as np
from IPython.display import display, Math, Latex # pro zobrazení matematických v

def ziskejDilciIntervaly(a, b, n):
    """
    Funkce vrací seznam dílčích intervalů ekvidistantního dělení intervalu [a,b]
    """
    dilciBody = np.linspace(a, b, n + 1)
    dilciIntervaly = []
    for i in range(n):
        dilciIntervaly.append((dilciBody[i], dilciBody[i + 1]))
    return dilciIntervaly


# testujme funkci ziskejDilciIntervaly
# a = 0
# b = 5
# n = 10
# dilciIntervaly = ziskejDilciIntervaly(a, b, n)
# print(dilciIntervaly)
```

```python
def ziskejRiemannuvIntegSoucet(f, a, b, n):
    """
    Funkce vypočítá aproximaci integrálu funkce f na intervalu [a,b] pomocí
    Riemannova integrálního součtu. Funkce f je zadána jako parametr funkce,
    interval [a,b] je zadán parametry a a b, počet dílčích intervalů ekvidistant
    dělení intervalu [a,b] je zadán parametrem n.
    """
    dilciIntervaly = ziskejDilciIntervaly(a, b, n)
    print(dilciIntervaly)
    soucet = 0
    for interval in dilciIntervaly:
        # vypočtěme střed intervalu a hodnotu funkce v tomto bodě
        stredIntervalu = (interval[0] + interval[1]) / 2
        # evaluujeme funkci f v bodě stredIntervalu symbolicky ne numericky (tj.
        hodnotaFunkce = f.subs(x, stredIntervalu)
        print(f"hodnota funkce v bodě {stredIntervalu} je {hodnotaFunkce}")
        # vypočtěme délku intervalu
        delkaIntervalu = interval[1] - interval[0]
        # přičteme k součtu hodnotu funkce v bodě středu intervalu násobenou dél
        soucet += hodnotaFunkce * delkaIntervalu
    # print(f"Riemannův integrální součet je {soucet}")
    display(Math(r"\sigma(%s) =  %g" %(sp.latex(f), soucet)))
    return soucet


# testujme funkci ziskejRiemannuvIntegSoucet
f = sp.sin(x)
a = 0
b = 5
n = 10
ziskejRiemannuvIntegSoucet(f, a, b, n)
```

```
[(0.0, 0.5), (0.5, 1.0), (1.0, 1.5), (1.5, 2.0), (2.0, 2.5), (2.5, 3.0), (3.0, 3.
5), (3.5, 4.0), (4.0, 4.5), (4.5, 5.0)]
hodnota funkce v bodě 0.25 je 0.247403959254523
hodnota funkce v bodě 0.75 je 0.681638760023334
hodnota funkce v bodě 1.25 je 0.948984619355586
hodnota funkce v bodě 1.75 je 0.983985946873937
hodnota funkce v bodě 2.25 je 0.778073196887921
hodnota funkce v bodě 2.75 je 0.381660992052332
hodnota funkce v bodě 3.25 je -0.108195134530108
hodnota funkce v bodě 3.75 je -0.571561318742344
hodnota funkce v bodě 4.25 je -0.894989358228583
hodnota funkce v bodě 4.75 je -0.999292788975378
```

$\sigma(\sin(x)) = 0.723854$

Out[ ]: 0.72385443698561

Example. Let A be the content of the area of the figure bounded by the graph of the function $f(x) = e^{-x}$ within the limits $x = 0$ and $x = 2$

(a) We express A as the limit of Riemann intergal sums and where the sampling points will be right endpoints of subintervals. Let's not count this limit.

(b) Let's estimate A using a Riemann integral sum, where the sample points will be the midpoints of the subintervals. Let's use 10 subintervals.

Solution. (a) $A = \int_0^2 e^{-x}dx = \lim_{n\to\infty}\sum_{i=1}^{n} e^{-x_i}\Delta x_i$ where $x_i = \frac{2i}{n}$ and $\Delta x_i = \frac{2}{n}$

Now let's substitute $\frac{2i}{n}$ for x_i in the sum above:

$$A = \lim_{n\to\infty}\sum_{i=1}^{n} e^{-\frac{2i}{n}}\frac{2}{n} =$$

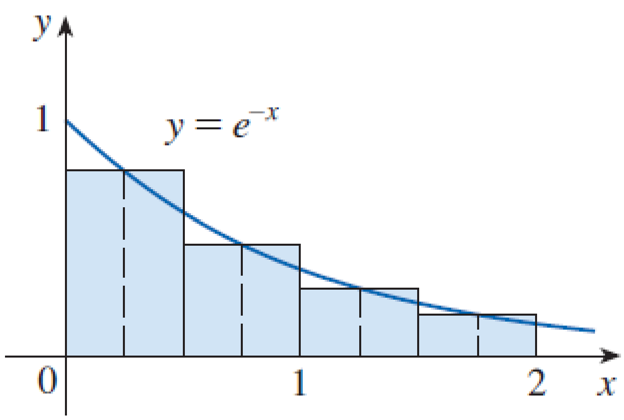
Let's break down the last sum:

$$= \lim_{n\to\infty}\left[e^{-2/n}\left(\frac{2}{n}\right) + e^{-4/n}\left(\frac{2}{n}\right) + \cdots + e^{-2(n)/n}\left(\frac{2}{n}\right)\right].$$

After editing, we get a limit in the form:

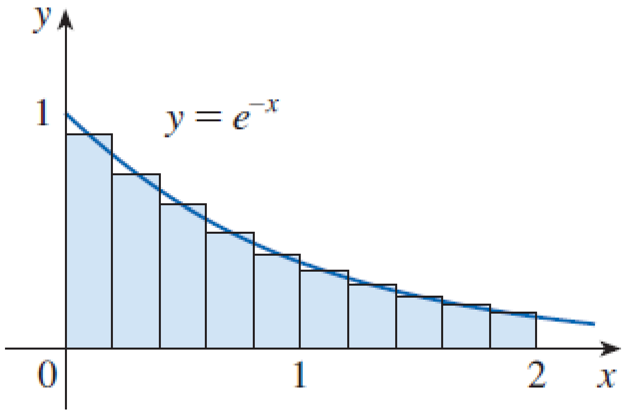
$$= \lim_{n\to\infty}\frac{2}{n}\left[e^{-2/n} + e^{-4/n} + \cdots + e^{-2(n)/n}\right] = \int_0^2 e^{-x}dx.$$

In the figure below we see the graph of the function e^{-x} in the interval $[0, 2]$ for four subintervals.



(b) We calculate the Riemann integral sum for 10 subintervals. This time the selection points will be the midpoints of the intervals, i.e. $x_i = 0.1 + 0.2i$ for $i = 0, 1, \ldots, 9$. The result is

$$M_{10} = \sum_{i=1}^{10} f(x_i)\cdot\Delta x = f(0.1)\Delta x + f(0.3)\Delta x + \cdots + f(1.9)\Delta x = \ = 0.2(e^{-0.1} + e^{-0.3} + e^{-0.5}\ldots + e^{-1.9})$$



Definite integral

Introduction

Definition. Let f be a function on the interval $\langle a, b\rangle$. Next, consider dividing the interval $\langle a, b\rangle$ into n subintervals of length $\Delta x = \frac{b-a}{n}$. Let $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, etc. and $x_n = b$.

Points $x_1^*, x_2^*, \dots, x_n^*$ are so-called **selection points**. Applies to: $x_i^* \in \langle x_{i-1}, x_i\rangle$ for $i = 1, 2, \dots, n$.

The definite integral is calculated as a limit:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that said limit exists and does not depend on the choice of sampling points. Then we say that the function f is **integrable** on the interval $\langle a, b\rangle$.

Note. The meaning of the mentioned limit can be precisely defined as follows: For every $\varepsilon > 0$ there is a N_ε such that for every $n > N_\varepsilon$:

$$\left|\sum_{i=1}^{n} f(x_i^*)\Delta x - \int_a^b f(x)\,dx\right| < \varepsilon$$

for any selection of $x_i^* \in \langle x_{i-1}, x_i\rangle$. sample points.

Theorem. If the function f is integrable on the interval $\langle a, b\rangle$, then the integral on this interval is uniquely given.

Theorem. If the function f is integrable on the interval $\langle a, b\rangle$, then it is a bounded function on the interval $\langle a, b\rangle$.

Note. Previous cannot be reversed. This means that there are functions on the interval $\langle, b\rangle$ that are bounded but do not have a Riemann integral from a to b.

An example can be the so-called Dirichlet function.

Historical note on the definition of integral. Integral sum in the form

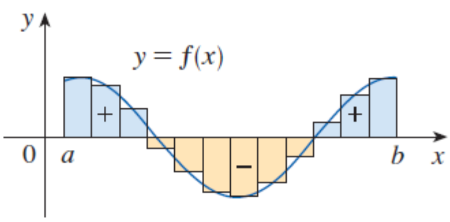
$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

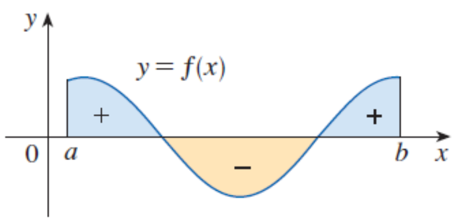
was introduced by the German mathematician Bernhard Riemann (1826-1866) in 1859. Bernhard Riemann received his Ph.D under the leadership of the legendary Gauss at the University of Göttingen. and stayed there to teach. gauss, who was not in the habit of praising other mathematics, talked about Riemann's "creative, active, truly spirited work". Also about the mathematical mind, awesome prolific and original. Definition of the integral we use is the merit Riemann. He also contributed significantly to the theory of functions and complex variables, mathematical physics, number theory and basics geometry. Riemann wide shot concept of space and geometry proved to be the right environment, 50 years later for Einstein's general theory. theory of relativity. Riemann's health was bad all his life and died of tuberculosis at the age of 39.

We already know that if the function f is positive on the interval $[a, b]$, then its integral is equal to of the area under the graph of the function f on the interval $[a, b]$. If the function f acquires both positive and negative values on the interval $[a, b]$, then its integral is equal to the difference of the content of the area under the graph of the function f and of the content of the area above the graph of the function f on the interval $[a, b]$. I mean

$$\int_a^b f(x)\, dx = A_1 - A_2,$$

where A_1 is the content of the area above the graph of the function f and A_2 is the content of the area below the graph of the function f.





Example: We calculate the Riemann integral sum for the function $f(x) = x^3 - 6x$, where x is in the interval $\langle 0, 3\rangle$. Let $n = 6$ and use the selection points that will be the right endpoints of the subintervals.

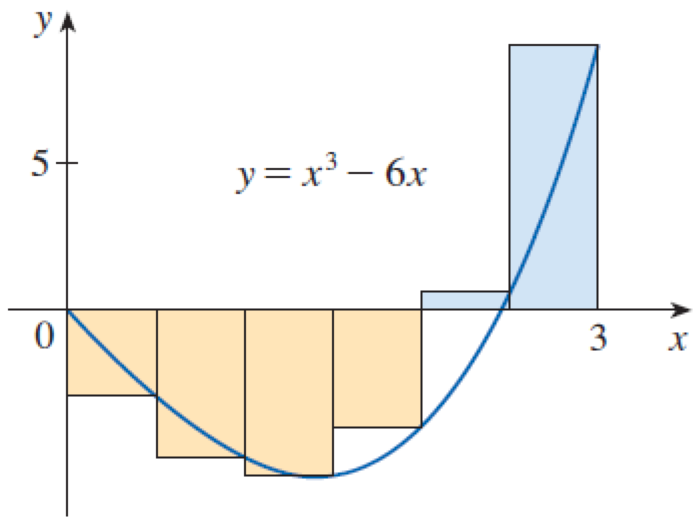
Solution: If n = 6, then $\Delta x = \frac{3}{6} = \frac{1}{2}$. So the selection points are

$x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 2, x_5 = \frac{5}{2}, x_6 = 3.$

So the Riemann integral sum for the function $f(x) = x^3 - 6x$, where $x \in [0, 3]$ is:

$$\begin{aligned} R_6 &= \sum_{i=1}^{6} f(x_i)\Delta x \\ &= f(0.5)\Delta x + f(1)\Delta x + f(1.5)\Delta x + f(2)\Delta x + f(2.5)\Delta x + f(3)\Delta x \\ &= \frac{1}{2}(-2.875 - 5 - 5.625 - 4 + 0.625 + 9) \\ &= -3.9375. \end{aligned}$$

Note that the value of the Riemann integral sum is negative. It is negative because the value of the integral sum is the sum of the contents of the blue rectangles minus the sum of the contents of the yellow rectangles. See image below.



Theorem (Sufficient condition for integrability). If the function $f: \langle a, b\rangle \to \mathrm{R}$ is continuous on the interval $\langle a, b\rangle$, then the function f is integrable on $\langle a, b\rangle$, i.e. the definite integral $\int_a^b f(x)\,dx$ exists.

Note. However, the opposite statement does not apply. This means that the existence of a definite integral $\int_a^b f(x)\,dx$ does not imply the continuity of the function f.

Theorem. If the function $f: \langle a, b\rangle \to \mathrm{R}$ is integrable on $\langle a, b\rangle$, then:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Example. Let's calculate the definite integral $\int_0^3 (x^3 - 6x)\,dx$ by definition.

Solution. We divide the interval $\langle 0, 3\rangle$ into n subintervals of length $\Delta x = \frac{3-0}{n} = \frac{3}{n}$ and count the values of $f(x_i)$ at the ends of these subintervals. Then we calculate the sum of these values and multiply it by the length of Δx.

To express the integral sum, we will use the right end points of the subintervals, i.e. $x_i = 0 + i\Delta x = i\Delta x$. Then we get:

$$\int_0^3 (x^3 - 6x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right]$$

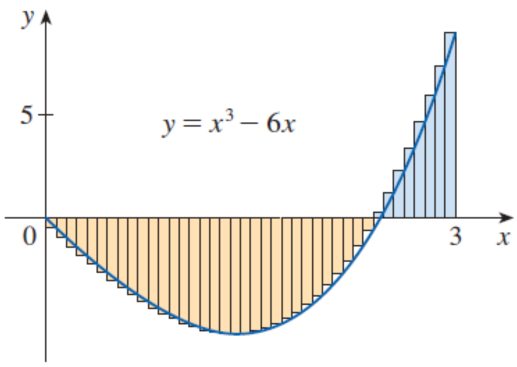
$$= \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\frac{27}{n^3}i^3 - \frac{18}{n}i\right]$$

$$= \lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i\right]$$

$$= \lim_{n\to\infty}\left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\frac{n(n+1)}{2}\right\}$$

$$= \lim_{n\to\infty}\left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)\right]$$

$$= \frac{81}{4} - 27 = -\frac{27}{4} = -6.75.$$



n	R_n
40	−6.3998
100	−6.6130
500	−6.7229
1000	−6.7365
5000	−6.7473

Basic properties of a definite integral

Let us assume that the integral $\int_a^b f(x)dx$ was defined under the assumption that $a < b.$ is valid. However, the definition also makes sense for the case when $a > b.$ Thus, we will define the integral $\int_b^a f(x)dx$ by the relation:

$$\int_b^a f(x)dx = -\int_a^b f(x)dx.$$

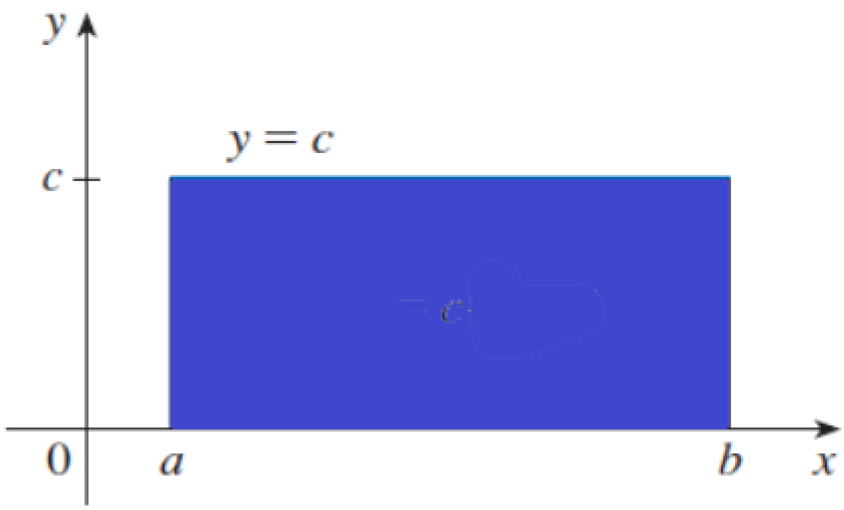
Next, we define for the case where $a = b$ is:

$$\int_a^a f(x)dx = 0.$$

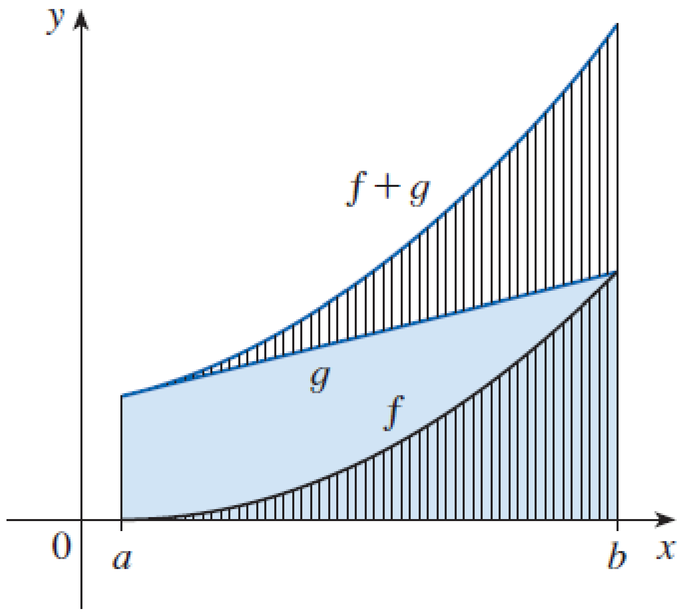
Sentence.

1. $\int_a^b c\,dx = c(b-a)$ where c is an arbitrary constant.
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$
4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$

Note. The integral in point 1. expresses the content of the rectangle with sides c and $b - a$. See picture.



The following figure illustrates property 2. from the previous sentence:



The additivity of the integral (Property 2) can be derived as follows:

$$\int_a^b [f(x) + g(x)]\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) + g(x_i)]\Delta x$$

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} f(x_i)\Delta x + \sum_{i=1}^{n} g(x_i)\Delta x \right]$$

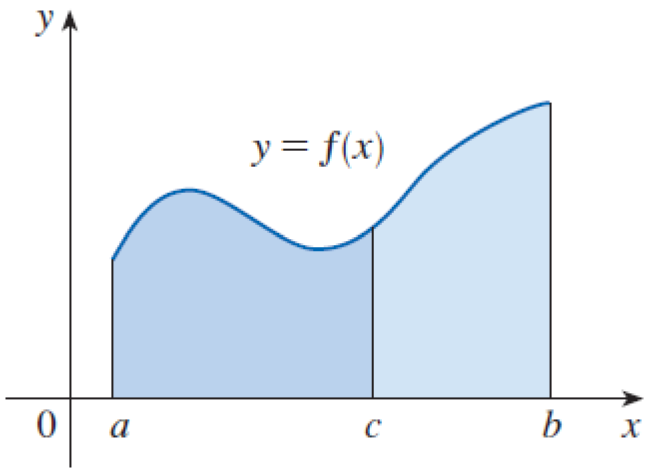
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x + \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\Delta x$$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Sentence.

5.

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$



Theorem. If the function $f: \langle a, b \rangle \to \mathrm{R}$ has a definite integral $\int_a^b f(x)\, dx$, then if $\langle c, d \rangle \subset \langle a, b \rangle$, then there is also an integral $\int_c^d f(x)\, dx$.

Now let's generalize the previous sentence:

Theorem. Let $f: \langle a, b \rangle \to \mathrm{R}$ be a function that has a definite integral $\int_a^b f(x)\, dx$. Further, let $\{\alpha, \beta, \gamma\} \subset \langle a, b \rangle$ be any set of real numbers. Then:

$$\int_\alpha^\beta f(x)\, dx = \int_\alpha^\gamma f(x)\, dx + \int_\gamma^\beta f(x)\, dx,$$

where the existence of any two mentioned integrals implies the existence of a third of the integrals and the mentioned equality.

Example. Suppose we know the integrals $\int_0^{10} f(x)dx = 17$ and $\int_0^8 f(x)dx = 12$. Then we find the value of the integral $\int_8^{10} f(x)dx$.

Solution. As a result of property 5, we find:

$$\int_0^8 f(x)dx + \int_8^{10} f(x)dx = \int_0^{10} f(x)dx. \int_8^{10} f(x)dx = \int_0^{10} f(x)dx - \int_0^8 f(x)dx = 17 - 12 = 5.$$

Sentence.

6. If f is integrable on the interval $[a, b]$, $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)dx \geq 0$.
7. If the functions f and g are integrable on the interval $[a, b]$, $f(x) \geq g(x)$ on $[a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
8. If f is integrable on the interval $[a, b]$, $m \leq f(x) \leq M$ on $[a, b]$, then

$$m(b - a) \leq \int_a^b f(x)dx \leq M(b - a).$$

9. If f is integrable on the interval $[a, b]$, then

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|\, dx.$$

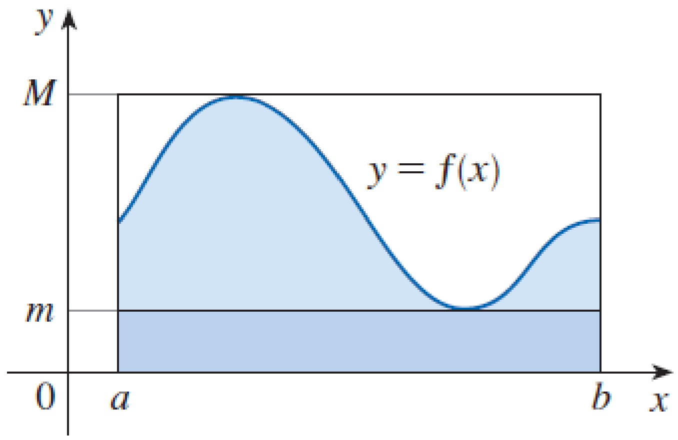
Ad 7.) If on the interval $\langle a, b\rangle$: $f(x) \geq g(x)$, then also holds that $f(x) - g(x) \geq 0$ on the interval $\langle a, b\rangle$. It follows from property (6) that

$$0 \leq \int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

It follows then that

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx.$$

Note. Property 8 is illustrated in the image below.



Example: Let's use property 8. and estimate the value of the integral:

$$\int_0^1 e^{-x^2} dx.$$

Solution: The function in the integrand $f(x) = e^{-x^2}$ is decreasing in the interval $[0, 1]$, the absolute maximum is $M = f(0) = e^0 = 1$. The absolute minimum is $m = f(1) = e^{-1}$. We use property 8. and estimate the value of the integral:

$$e^{-1}(1-0) \leq \int_0^1 e^{-x^2} dx \leq e^0(1-0)$$
$$e^{-1} \leq \int_0^1 e^{-x^2} dx \leq e^0 = 1.$$

Since it is $e^{-1} \approx 0.3679$ and $e^0 = 1$, the estimate of the value of the integral is $0.3679 \leq \int_0^1 e^{-x^2} dx \leq 1.$

Sufficient conditions for the existence of an integral

Cauchy integrability criterion

Theorem (Cauchy integrability criterion): Let $f: \langle a, b\rangle \to \mathrm{R}$ be a bounded function. A function f is Riemann integrable if and only if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for any two divisions P_1 and P_2 of the interval $\langle a, b\rangle$ such that the length of the subintervals P_1 and P_2 is less than δ,

$$\left|\sum_{i=1}^{n} f(x_i^*)\Delta x_i - \sum_{i=1}^{n} f(y_i^*)\Delta y_i\right| < \varepsilon,$$

where x_i^* and y_i^* are arbitrary points in the subintervals $\langle x_{i-1}, x_i\rangle$ and $\langle y_{i-1}, y_i\rangle$ of the respective divisions P_1 and P_2. Here x_i or y_i are the ends of the subintervals of the division P_1 or P_2 and $\Delta x_i = x_i - x_{i-1}$ respectively $\Delta y_i = y_i - y_{i-1}$ are the lengths of the sub-intervals of the division P_1 or P_2.

Theorem. If $f: \langle a, b\rangle \to \mathrm{R}$ is a continuous function on the interval $\langle a, b\rangle$, then there is an integral $\int_a^b f(x)\,dx$.

Theorem. If $f: \langle a, b\rangle \to \mathrm{R}$ is a monotonic function on the interval $\langle a, b\rangle$, then there exists an integral $\int_a^b f(x)\,dx$.

Definition. We say that a **set $N \subset \mathbf{R}$ has (Lebesgue) measure zero** if for every $\varepsilon > 0$ there exists a sequence of $\{I_n\}_{n=1}^{\infty}$ intervals such that $N \subset \bigcup_{n=1}^{\infty} I_n$ and $\sum_{n=1}^{\infty} |I_n| < \varepsilon$. Here $|I_n|$ denotes the length of the I_n interval.

Lebesgue criterion for the existence of an integral

Theorem (Lebesgue Criterion). A bounded function $f: \langle a, b\rangle \to \mathrm{R}$ has a Riemann definite integral $\int_a^b f(x)\,dx$ if and only if the set of points where the function is discontinuous has measure zero.

Example. Consider the function $f: \langle 0, 1\rangle \to \mathrm{R}$ defined by the rule

$$f(x) = \begin{cases} \frac{1}{n} & \text{pro } x = \frac{1}{n}, n \in \mathrm{N}, \\ 0 & \text{jinak.} \end{cases}$$

The set of discontinuity points is $N = \{1/n \mid n \in \mathrm{N}\}$, which has measure zero. So the function f has a Riemannian definite integral.

Fundamental theorem of integral calculus, first part

Theorem (Newton-Leibniz formula). Suppose that f and F are two functions on the interval $\langle a, b\rangle$ such that the following conditions hold:

(a) F is a continuous function on the interval $\langle a, b\rangle$,

(b) $F'(x) = f(x)$ for all $x \in (a, b)$,

(c) f is integrable on the interval $\langle a, b\rangle$.

Then:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof. Let $\varepsilon > 0$ and let there exist a definite integral $\int_a^b f(x)\,dx = I$. Then there exists $N_\varepsilon \in \mathrm{N}$ such that for all $n \geq N_\varepsilon$:

$$\left| \sum_{i=1}^{n} f(x_i^*)\Delta x - \int_a^b f(x)\,dx \right| < \varepsilon$$

for any selection of selection points $x_i^* \in \langle x_{i-1}, x_i \rangle$.

If we apply Lagrange's mean value theorem on each $\langle x_{i-1}, x_i \rangle$ interval, we get:

$$F(x_i) - F(x_{i-1}) = F'(x_i^*) \cdot (x_i - x_{i-1}) \text{ pro všechna } i = 1, 2, \ldots, n.$$

Therefore, for every $i = 1, 2, \ldots, n$: $x_i^* \in (x_{i-1}, x_i)$. Since $F'(x) = f(x)$ for all $x \in (a, b)$, we get: $F'(x_i^*) = f(x_i^*)$ for all $i = 1, 2, \ldots, n$.

Summing all the equations we get:

$$\begin{aligned} F(b) - F(a) &= \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) \\ &= \sum_{i=1}^{n} F'(x_i^*) \cdot (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} f(x_i^*) \cdot (x_i - x_{i-1}). \end{aligned}$$

If we now substitute $F(b) - F(a)$ for $\sum_{i=1}^{n} f(x_i^*)\Delta x$ in the relation (1), we get:

$$\left| F(b) - F(a) - \int_a^b f(x)\,dx \right| < \varepsilon.$$

Since we chose $\varepsilon > 0$ arbitrarily at the beginning, we thus proved the existence of the integral $\int_a^b f(x)\,dx$ and at the same time that

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

So the proof is complete. $\square$

Example: Let's show that $\int_0^1 x^2\,dx = \frac{1}{3}$.

Solution: The function $f(x) = x^2$ is integrable on the interval $\langle 0, 1 \rangle$. Next, consider the function $F(x) = \frac{1}{3}x^3$. The function F is continuous on the interval $\langle 0, 1 \rangle$ and $F'(x) = f(x)$ for all $x \in (0, 1)$. Therefore:

$$\int_0^1 x^2\, dx = F(1) - F(0) = \frac{1}{3} - 0 = \frac{1}{3}.$$

Example. Consider the function $H(x) = 2\sqrt{x}$ on the interval $\langle 0, 1\rangle$. The function H is continuous on the interval $\langle 0, 1\rangle$ and $H'(x) = 1/\sqrt{x}$ for all $x \in (0, 1)$. Furthermore, consider the function $h\colon \langle 0, 1\rangle \to \mathrm{R}$ such that $h(x) = 1/\sqrt{x}$ for all $x \in (0, 1\rangle$ and $h(0) = 0$. Then the function h is not integrable on the interval $\langle 0, 1\rangle$ neboť zde není omezenou funkcí. Therefore, the fundamental theorem of integral calculus cannot be used in this case.

If we were to substitute into the so-called Newton-Leibniz formula, then on the right-hand side we would get:

$$H(1) - H(0) = 2\sqrt{1} - 2\sqrt{0} = 2.$$

Fundamental theorem of integral calculus, second part

Theorem. Let the function $f\colon \langle a, b\rangle \to \mathrm{R}$ have a Riemann definite integral $\int_a^b f(x)\, dx$. Let F be a function defined on the interval $\langle a, b\rangle$ by the prescription

$$F(x) = \int_a^x f(t)\, dt.$$

Then the function F is continuous on the interval $\langle a, b\rangle$ and in addition it holds that if for every $x \in \langle a, b\rangle$ there is $|f(x)| \leq M,$ then the function F is Lipschitzov continuous on the interval $\langle a, b\rangle$ with constant M. That is. for each $s, t \in \langle a, b\rangle$:

$$|F(s) - F(t)| \leq M|s - t|.$$

Theorem. Assume that f is integrable on the interval $\langle a, b\rangle$ and let the function f be continuous at the point $c \in \langle a, b\rangle$. Then for every $t \in \langle a, b\rangle$ let us put $F(x) = \int_a^x f(t)\, dt$. Then the function $F\colon \langle a, b\rangle \to \mathrm{R}$ is differentiable at the point c and it holds:

$$F'(c) = f(c).$$

Proof. Let $c \in (a, b)$ and let $\varepsilon > 0$. Then there exists $\eta_\varepsilon > 0$ such that for all $x \in \langle c, c + \eta_\varepsilon\rangle$:

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon.$$

Now let h meet $0 < h < \eta_\varepsilon$. Then it follows from the so-called interval additivity:

$$F(c + h) = \int_a^{c+h} f(x)\, dx = \int_a^c f(x)\, dx + \int_c^{c+h} f(x)\, dx = F(c) + \int_c^{c+h} f(x)\, dx.$$

It follows from there:

$$F(c + h) - F(c) = \int_c^{c+h} f(x)\, dx.$$

Therefore, on the interval $\langle c, c+h\rangle$:

$$(f(c)-\varepsilon)\cdot h \le F(c+h)-F(c)=\int_c^{c+h} f(x)\,dx \le (f(c)+\varepsilon)\cdot h.$$

Dividing the previous $h>0$ inequalities, we get:

$$\left|\frac{F(c+h)-F(c)}{h}-f(c)\right| \le \varepsilon.$$

Now from the definition of limits from the right it follows:

$$\lim_{h\to 0+}\frac{F(c+h)-F(c)}{h}=f(c).$$

Similarly, it can be proved for the limit from the left that

$$\lim_{h\to 0-}\frac{F(c+h)-F(c)}{h}=f(c).$$

Therefore:

$$F'(c)=\lim_{h\to 0+}\frac{F(c+h)-F(c)}{h}=\lim_{h\to 0-}\frac{F(c+h)-F(c)}{h}=f(c).$$

So the proof is complete. $\square$

Corollary. If f is a continuous function on the interval (a,b), then to the function f there exists on the interval (a,b) a primitive function F such that $F'(x)=f(x)$ for all $x\in(a,b)$.

The following example shows that the preceding corollary gives a sufficient and not a necessary condition for the existence of a primitive function F for a function f on the interval (a,b).

Example. A classic example of a differentiable function with a discontinuous derivative is the function:

$$f(x)=\begin{cases} x^2\sin(\frac{1}{x}), & \text{if } x\neq 0,\\ 0, & \text{if } x=0.\end{cases}$$

This function is defined, for example, on the open interval $(-1,1)$. Let's see why this function is differentiable everywhere, but its derivative is discontinuous at $x=0$. For $x\neq 0$ we can calculate the derivative using the product rule and the chain rule:

$$f'(x)=2x\sin(\frac{1}{x})+x^2\cos(\frac{1}{x})\cdot(-\frac{1}{x^2})=2x\sin(\frac{1}{x})-\cos(\frac{1}{x}).$$

The derivative clearly exists for $x\neq 0$ and is continuous everywhere except $x=0$. Now consider $x = 0$. We use the definition of derivative to see if there is:

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin(\frac{1}{h})}{h} = \lim_{h \to 0} h \sin(\frac{1}{h}).$$

Since $|\sin(\frac{1}{h})| \le 1$ for all $h \ne 0$, we have $|h \sin(\frac{1}{h})| \le |h|$. When $h \to 0$, we get $h \sin(\frac{1}{h}) \to 0$, so the limit exists and $f'(0) = 0$. So the function is differentiable everywhere. However, the derivative is discontinuous at the point $x = 0$. To see this, consider the limit for $x \to 0$ the derivative for $x \ne 0$:

$$\lim_{x \to 0} (2x \sin(\frac{1}{x}) + \cos(\frac{1}{x})).$$

The second term, $\cos(\frac{1}{x})$, oscillates between -1 and 1 when $x \to 0$, and the first term, $2x \sin(\frac{1}{x})$, approaches 0 when $x \to 0$. The limit does not exist when $x \to 0$, so the derivative $f'(x)$ is discontinuous for $x = 0$.

Let us show in more detail why the previous limit (1) does not exist. We want to show that the limit:

$$\lim_{x \to 0} (2x \sin(\frac{1}{x}) + \cos(\frac{1}{x}))$$

does not exist We will show that for a particular choice ϵ we cannot find a suitable δ.

Let's choose $\epsilon = \frac{1}{2}$. Assume that the limit exists and is equal to L. Then there should exist a $\delta > 0$ such that for all x with $0 < |x| < \delta$ we have:

$$\left| 2x \sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right) - L \right| < \epsilon = \frac{1}{2}.$$

Now consider two sequences of points:

1. $x_n = \frac{1}{2\pi n}$ where $n \in \mathbb{N}$. In this case we have:

$$2x_n \sin\left(\frac{1}{x_n}\right) + \cos\left(\frac{1}{x_n}\right) = 2\left(\frac{1}{2\pi n}\right)\sin(2\pi n) + \cos(2\pi n) = 1.$$

2. $y_n = \frac{1}{(2n+1)\pi}$ where $n \in \mathbb{N}$. In this case we have:

$$2y_n \sin\left(\frac{1}{y_n}\right) + \cos\left(\frac{1}{y_n}\right) = 2\left(\frac{1}{(2n+1)\pi}\right)\sin((2n+1)\pi) + \cos((2n+1)\pi) = -1.$$

Once $n \to \infty$, both sequences, x_n and y_n, converge to 0. For any $\delta > 0$, we can find a n large enough that $0 < |x_n| < \delta$ and $0 < |y_n| < \delta$. However, for these x_n and y_n:

$$\left|2x_n\sin\left(\frac{1}{x_n}\right)+\cos\left(\frac{1}{x_n}\right)-L\right| \geq |1-L|$$

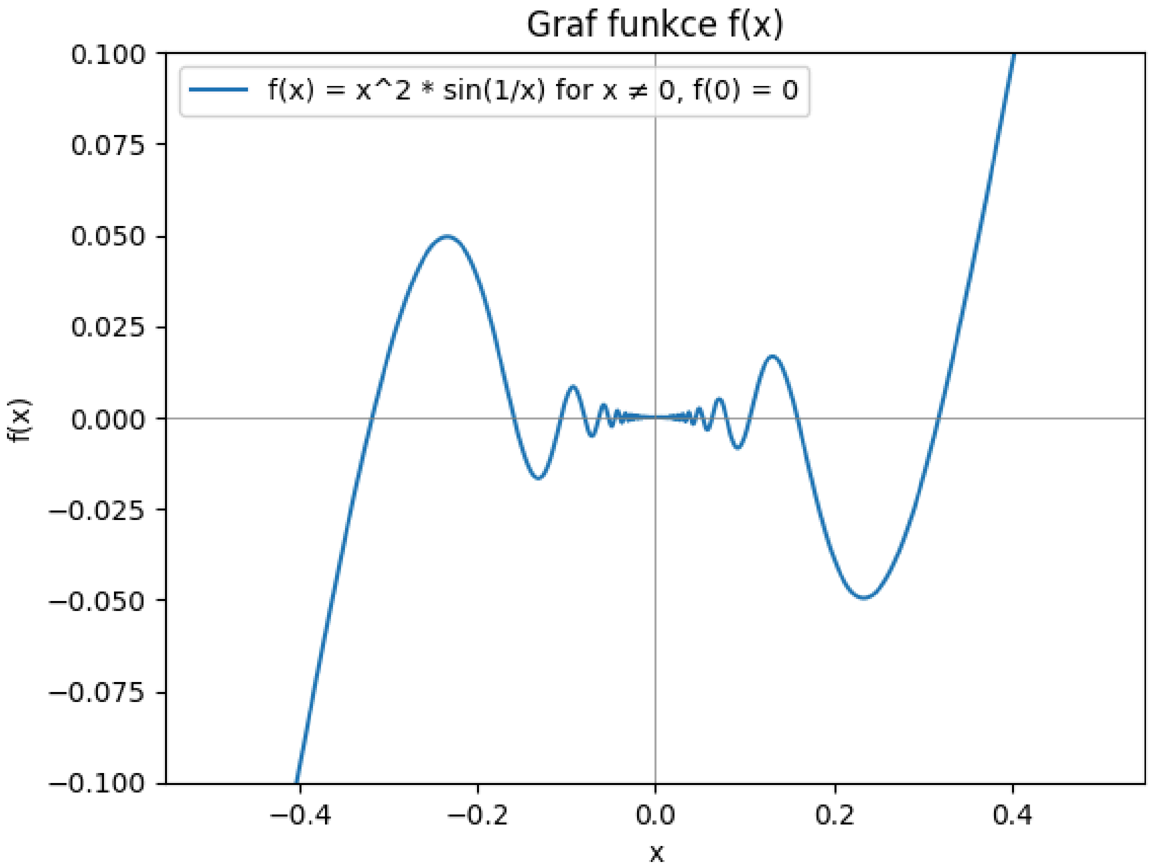
a

$$\left|2y_n\sin\left(\frac{1}{y_n}\right)+\cos\left(\frac{1}{y_n}\right)-L\right| \geq |-1-L|$$

Since we assumed that the limit exists, it should satisfy the inequality with $\epsilon = \frac{1}{2}$ for both x_n and y_n. But $|1-L|$ and $|-1-L|$ cannot both be less than $\frac{1}{2}$ because their sum is equal to 2. This leads to a dispute and we conclude that there is no limit.

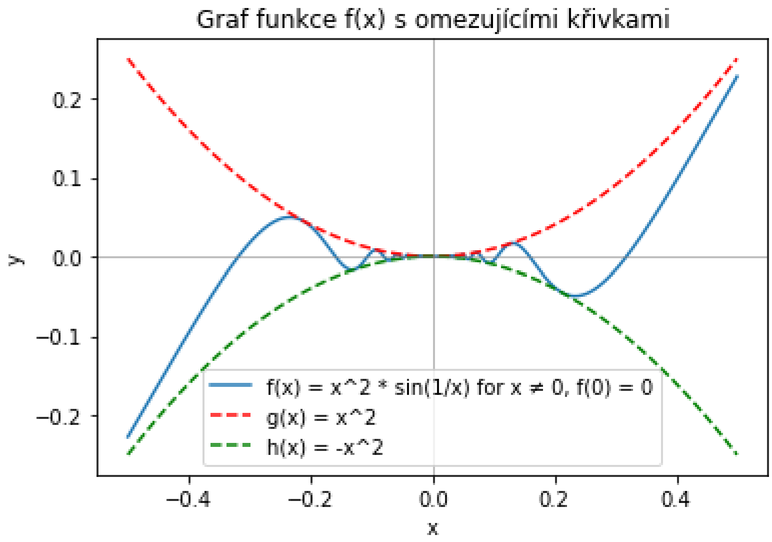
Therefore, the derivative $f'(x)$ is discontinuous for $x = 0$.

In [ ]:
```
#@title
import numpy as np
import matplotlib.pyplot as plt

def f(x):
    return x**2 * np.sin(1/x) if x != 0 else 0

x = np.linspace(-0.5, 0.5, 1000)
y = np.vectorize(f)(x)

plt.plot(x, y, label='f(x) = x^2 * sin(1/x) for x ≠ 0, f(0) = 0')
plt.axvline(0, color='gray', linewidth=0.5)
plt.axhline(0, color='gray', linewidth=0.5)
plt.xlabel('x')
plt.ylabel('f(x)')
plt.legend()
plt.title('Graf funkce f(x)')
# nastavme rozsah osy y
plt.ylim(-0.1, 0.1)
plt.show()
```



In [ ]:

```
#@title
import numpy as np
import matplotlib.pyplot as plt

def f(x):
    return x**2 * np.sin(1/x) if x != 0 else 0

def g(x):
    return x**2

def h(x):
    return -x**2

x = np.linspace(-0.5, 0.5, 1000)
y_f = np.vectorize(f)(x)  # vektorizace funkce f pro všechny prvky vektoru x
y_g = np.vectorize(g)(x)
y_h = np.vectorize(h)(x)

plt.plot(x, y_f, label='f(x) = x^2 * sin(1/x) for x ≠ 0, f(0) = 0')
plt.plot(x, y_g, linestyle='dashed', color='red', label='g(x) = x^2')
plt.plot(x, y_h, linestyle='dashed', color='green', label='h(x) = -x^2')
plt.axvline(0, color='gray', linewidth=0.5)
plt.axhline(0, color='gray', linewidth=0.5)
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.title('Graf funkce f(x) s omezujícími křivkami')
plt.show()
```



Example. Let's find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\,dt$.

Solution. The function $f(t) = \sqrt{1+t^2}$ is a continuous function on the interval $(-\infty, \infty)$. It follows from the fundamental theorem of integral calculus that for every x:

$$g'(x) = \sqrt{1+x^2}.$$

Example. Let's calculate the integral $\int_0^3 e^x\,dx$.

Solution. Let $f(x) = e^x$. Then f be a continuous function on the interval $\langle 0, 3 \rangle$. The function $F(x) = e^x$ is continuous on the interval $\langle 0, 3 \rangle$ and $F'(x) = f(x)$ for all $x \in (0, 3)$. Since f is integrable on the interval $\langle 0, 3 \rangle$, we can use the fundamental theorem of integral calculus.

It follows from the fundamental theorem of integral calculus that

$$\int_0^3 e^x\,dx = \int_0^3 f(x)\,dx = F(3) - F(0) = e^3 - 1.$$

Notation. The notation used for the expression $F(b) - F(a)$ is: $[F(x)]_a^b$. Ie.

$$[F(x)]_a^b = F(b) - F(a).$$

$$F(x)\,|_a^b = F(b) - F(a).$$

Example. Let's calculate the content of the figure bounded by the graph of the function $f(x) = x^2$ and the lines $x = 0$ and $x = 1$.

Solution. Primitive function to function f is function $F(x) = \frac{x^3}{3}$. Then the content of the figure is equal to

$$A = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

Example. Let's calculate the content of the figure bounded by the graph of the function $f(x) = \cos x$ and the lines $x = 0$ and $x = \pi/2$.

Solution. The primitive function to function f is function $F(x) = \sin x$. Then the content of the shape is equal to:

$$A = \int_0^{\pi/2} \cos x\,dx = [\sin x]_0^{\pi/2} = 1.$$

Indefinite integral

Definition. The symbol $\int f(x)\,dx$ denotes or is called the indefinite integral of the function f represents the general form of a primitive function to the function f. Thus, if F is a primitive function to the function f, then

$$\int f(x)\,dx = F(x) + C,$$

where C is an arbitrary constant.

Note. It is important to say that the (NI) formula only for x belonging to a certain interval!!! If, for example, we consider the function $f(x) = \frac{1}{x^2}$, then for the function

$$F(x) = \begin{cases} -\frac{1}{x} + C_1 & x < 0, \\ -\frac{1}{x} + C_2 & x > 0, \end{cases}$$

it holds that $F'(x) = \frac{1}{x^2}$, for all $x \neq 0$. So for every $x \neq 0$ the formula does not hold:

$$\int \frac{1}{x^2}\,dx = -\frac{1}{x} + C.$$

The previous relation only applies to $x \in (0, \infty)$ or $x \in (-\infty, 0)$.

Example. As we know, the function $F(x) = \frac{x^3}{3}$ is a primitive function to the function $f(x) = x^2$. Then it is

$$\int x^2\,dx = \frac{x^3}{3} + C.$$

Theorem (Basic Properties of the Indefinite Integral).

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

Substitution in the integral

Substitution in the indefinite integral

Theorem. If $u = g(x)$ is a differentiable function whose domain is a subset of the interval I and the function f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

Proof. The theorem is a consequence of the theorem on the derivative of a composite function. If $F(u)$ is a primitive function to $f(u)$ and $u = g(x)$ then

$$F'(u) = f(u) \qquad \text{a} \qquad (F(g(x)))' = f(g(x))g'(x).$$

Example. Let's calculate the $\int x^3\cos(x^4+2)\,dx$ integral.

Solution. Let's apply the $u = x^4 + 2$ substitution and get:

$$du = 4x^3\,dx$$

$$\int x^3\cos(x^4+2)\,dx = \int \cos u \cdot \frac{1}{4}\,du = \frac{1}{4}\int \cos u\,du$$

$$= \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4+2) + C.$$

Exercise. Verify the correctness of the result in the previous example by differentiation.

Example. Let's calculate the $\int \sqrt{2x+1}\,dx$ integral.

Solution. Let's apply the substitution and put: $u = 2x + 1$. Then we get:

$$du = 2\,dx$$

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u} \cdot \frac{1}{2}\,du = \frac{1}{2}\int \sqrt{u}\,du$$

$$= \frac{1}{2} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{1}{3}u^{\frac{3}{2}} + C = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C.$$

$$\int \sqrt{2x+1}\,dx = \frac{1}{2}\int \sqrt{2x+1} \cdot 2\,dx.$$

Example. Let's calculate the $\int \frac{x}{\sqrt{1-4x^2}}\,dx$ integral.

Solution. Let $u = 1 - 4x^2$. Then we get: $du = -8x\,dx$. So $xdx = -\frac{1}{8}du$. Let's apply the substitution and get:

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = -\frac{1}{8}\int \frac{1}{\sqrt{u}}\,du$$

$$= -\frac{1}{8}\int u^{-\frac{1}{2}}\,du = -\frac{1}{8} \cdot (2\sqrt{u}) + C = -\frac{1}{4} \cdot \sqrt{1-4x^2} + C.$$

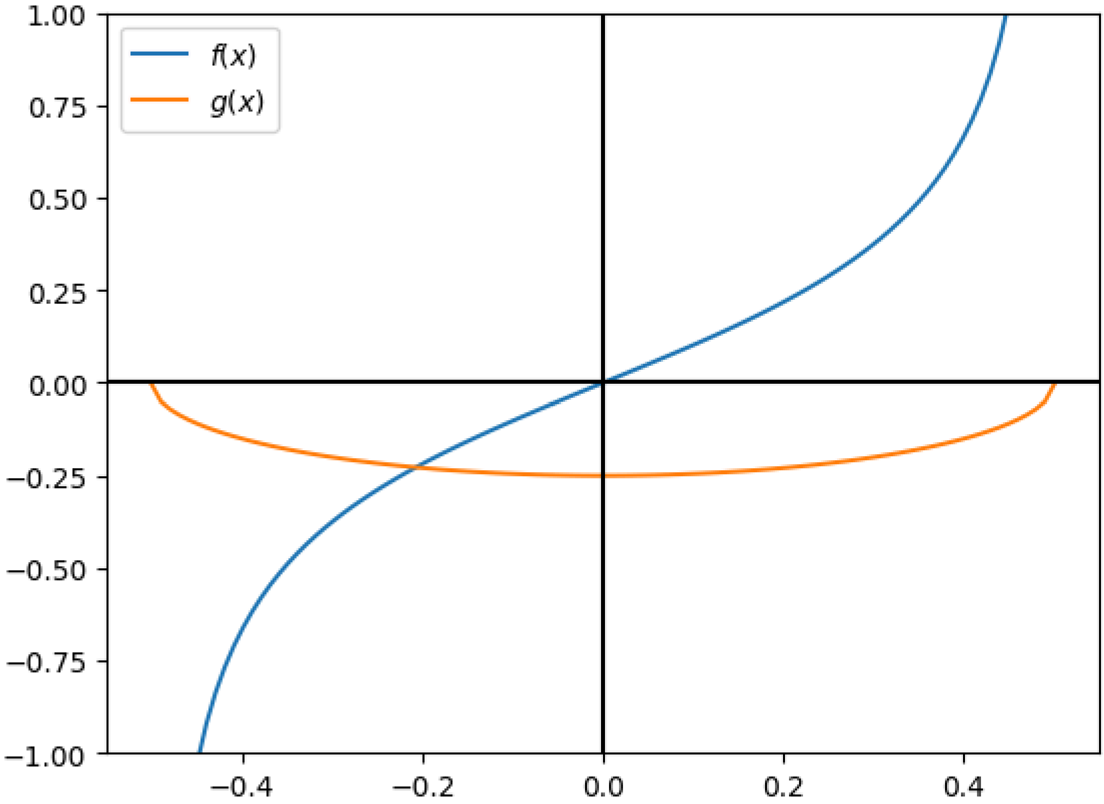
Let's now create an image of the graphs of the f and g functions and see how the two graphs are related to each other. Here $f(x) = \frac{x}{\sqrt{1-4x^2}}$ and $g(x) = \int f(x)\,dx$.

In [ ]:
```
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math, Latex

sp.init_printing(use_latex='mathjax')

x = sp.Symbol('x')
f = x / (sp.sqrt(1 - 4 * x**2))
g = sp.integrate(f, x)
display(Math(r"$f(x) = %s$" % sp.latex(f)))
display(Math(r"$g(x) = \int %s = %s$" % (sp.latex(f), sp.latex(g))))

a = -1/2 + 0.0001 # musíme přidat malé číslo, aby se nevyskytla chyba dělení nul
b = 1/2 - 0.0001

x_values = np.linspace(a, b, 100)  # vytvoříme 100 hodnot x v intervalu [a,b]

y1 = [f.subs(x, x1) for x1 in x_values]  # vypočítáme hodnoty funkce f(x) pro ka
y1 = np.array(y1)  # převedeme na numpy pole

y2 = [g.subs(x, x1) for x1 in x_values]
y2 = np.array(y2) # převedeme na numpy pole

plt.plot(x_values, y1, label=r"$f(x)$")  # r"" znamená, že řetězec je raw string
```

```
plt.plot(x_values, np.real(y2), label=r"$g(x)$") # np.real() je zde proto, aby s
plt.legend()
plt.ylim(-1, 1) # nastavíme rozsah osy y
# přidejme plot obou os
plt.axhline(y=0, color='k')
plt.axvline(x=0, color='k')

plt.show() # zobrazíme graf
```

$$f(x) = \frac{x}{\sqrt{1-4x^2}}$$

$$g(x) = \int \frac{x}{\sqrt{1-4x^2}} = -\frac{\sqrt{1-4x^2}}{4}$$



Substitution in the definite integral

Theorem. If the derivative g' is continuous on the interval $\langle a, b\rangle$ and the function f is continuous on the interval $I \supset \{u = g(x) : x \in \langle a, b\rangle\}$, then for $u = g(x)$:

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example. Let's calculate the $\int_1^4 \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ integral.

Solution. Let's apply the substitution $u = g(x) = \sqrt{x}$ where $x \in \langle 1, 4\rangle$. Then $g'(x) = \frac{1}{2\sqrt{x}}$ is continuous to $\langle 1, 4\rangle$. Next, consider the $f(u) = 2\sin u$ function. The function f is a continuous function on the interval $I = (-\infty, \infty)$ and $f(g(x)) \cdot g'(x) = \frac{\sin\sqrt{x}}{\sqrt{x}}$. Applying the theorem, we get:

$$\int_1^4 \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx = \int_1^4 2\sin\sqrt{x} \cdot \frac{1}{2\sqrt{x}}\, dx$$
$$= \int_1^2 2\sin u\, du = 2[-\cos u]_1^2 = 2[-\cos 2 + \cos 1].$$

Example. Let's calculate the $\int_1^e \frac{\ln x}{x}\, dx$ integral.

Solution. Let's apply the substitution $u = g(x) = \ln x$ where $x \in \langle 1, e\rangle$. Then $g'(x) = \frac{1}{x}$ is continuous to $\langle 1, e\rangle$. Next, consider the $f(u) = u$ function. The function f is a continuous function on the interval $I = (-\infty, \infty)$ and $f(g(x)) \cdot g'(x) = \frac{\ln x}{x}$. Applying the theorem, we get:

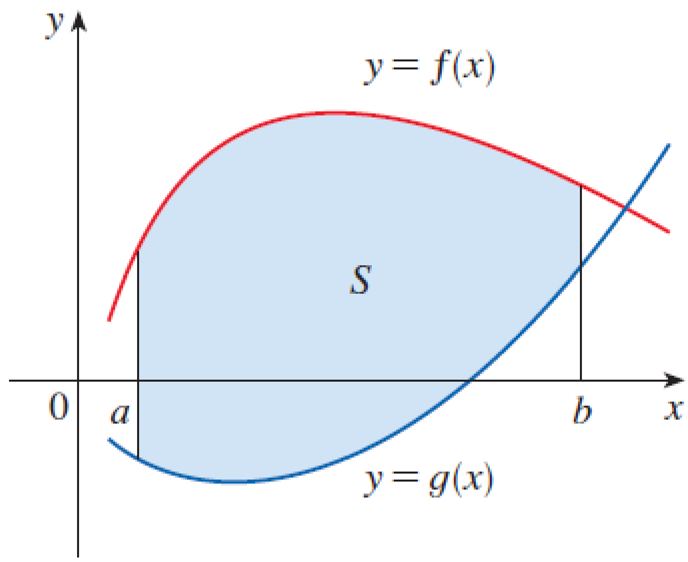
$$\int_1^e \frac{\ln x}{x}\, dx = \int_1^e \ln x \cdot \frac{1}{x}\, dx$$
$$= \int_0^1 u\, du = \left[\frac{1}{2}u^2\right]_0^1 = \frac{1}{2}.$$

Application of integral calculus

The content of the shape between the two curves

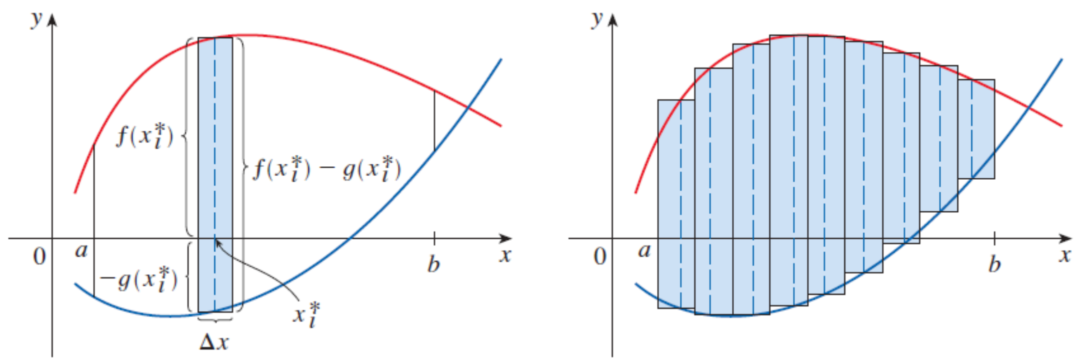
Content between two curves: integration with respect to x

Consider the figure S bounded by two curves $y = f(x)$ and $y = g(x)$ and where $x \in \langle a, b\rangle$. Assume $f(x) \geq g(x)$ for $x \in \langle a, b\rangle$ and both functions are continuous on $\langle a, b\rangle$.



The A content of the S shape can now be approximated using integral sums similar to the case of the shape bounded by the graph of the positive function f. I.e. in sum:

$$A \approx \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x.$$



It is now natural to define the content of the S shape as the limit:

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x.$$

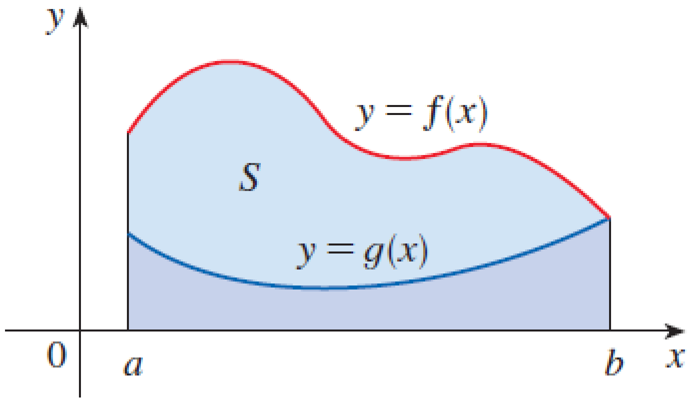
So the following theorem holds.

Theorem. The content of the A figure bounded by the curves $y = f(x)$, $y = g(x)$, the lines $x = a$, $x = b$ and where the functions f and g are continuous such that $f(x) \geq g(x)$ for each $x \in [a, b]$ is equal to:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

Note. If the functions f and g in the previous sentence are both positive, then A is equal to:

$$A = [\text{obsah obrazce pod grafem funkce } f] - [\text{obsah obrazce pod grafem funkce } g].$$
$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]\,dx.$$



Example. Let's derive the formula for the area of the circle: $A = \pi r^2$ using the integral.

Solution. We will use the relation $A = \int_{-r}^{r} \left[\sqrt{r^2 - x^2} - (-\sqrt{r^2 - x^2})\right] dx.$

I mean

$$A = 4 \cdot \int_0^r \sqrt{r^2 - x^2} dx$$

$$\int_0^r \sqrt{r^2 - x^2} dx = \int_0^r \sqrt{r^2(1 - (\frac{x}{r})^2)} dx$$

$$= r\int_0^r \sqrt{1 - (\frac{x}{r})^2} dx = \ldots$$

Now let's use the substitution: $u = g(x) = \frac{x}{r}$, so $du = \frac{dx}{r}$ and we get

$$= r^2 \int_0^1 \sqrt{1 - u^2} du = \ldots$$

Consider the indefinite integral $\int \sqrt{1 - u^2} du$.

Let's put $u = g(x) = \sin x$ where $x \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$. Then after differentiating we get:

$$du = \cos x dx.$$

Next

$$\sqrt{1 - u^2} = \sqrt{1 - \sin^2 x} = \sqrt{\cos^2 x} = \cos x = f(g(x)).$$

$$g'(x) = \cos x.$$

$$\int f(u)du = \int \sqrt{1-u^2}du = \int f(g(x))g'(x)dx = \int \cos x \cos x dx = \dots$$

$$\dots = \int \cos^2 x dx =$$

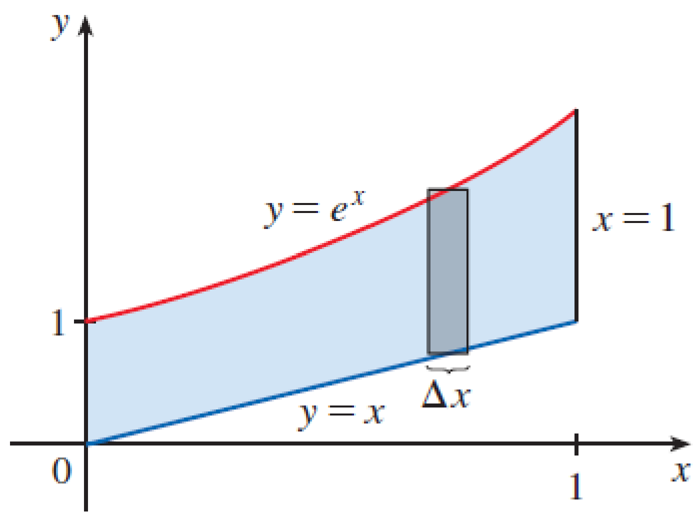
$$= \frac{1}{2}\int (1+\cos 2x)dx = \dots$$

$$= \frac{1}{2} \cdot (x + \frac{1}{2}\sin 2x) + C, \quad x = \arcsin u, \quad u \in \langle -1, 1 \rangle$$

$$\int_0^1 \sqrt{1-u^2}du = \left[\frac{1}{2} \cdot (\arcsin u + \frac{1}{2}\sin(2 \cdot \arcsin u))\right]_0^1 = \frac{\pi}{4}.$$

Example. Let's calculate the content of the shape S bounded by the curves $y = e^x$ and $y = x$, with $x \in \langle 0, 1 \rangle$.

Solution. The shape S is shown in the figure:



Obviously, the upper curve is the $y = e^x$ curve and the lower curve is the $y = x$ curve. So $f(x) = e^x$ and $g(x) = x$, $a = 0$ and $b = 1$.

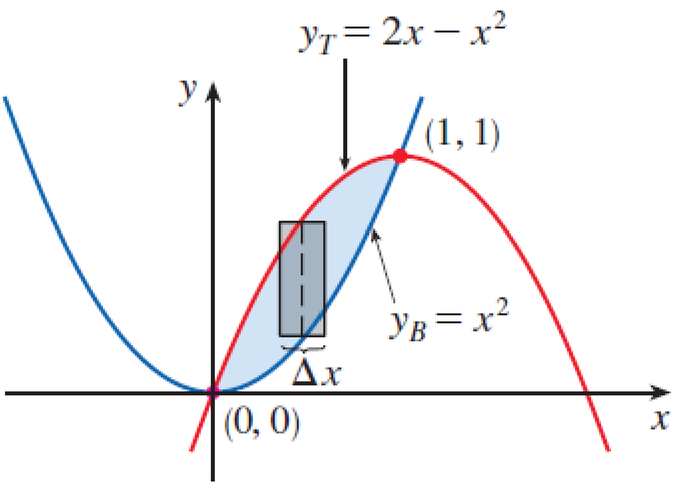
$$A = \int_0^1 (e^x - x)\, dx = \int_0^1 e^x\, dx - \int_0^1 x\, dx =$$

$$= \left[e^x\right]_0^1 - \left[\frac{x^2}{2}\right]_0^1 =$$

$$= e^1 - 1 - \frac{1}{2} = e - 1.5.$$

Example. Let's calculate the content of the A shape that is bounded by the $y = x^2$ curve and the $y = 2x - x^2$ curve.

Solution. Let's show a picture of curves and shapes.



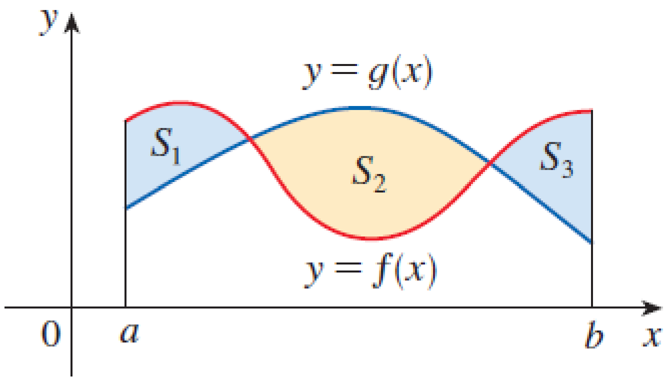
It can be seen from the figure that the upper curve is the $y_T = 2x - x^2$ curve and the lower curve is the $y_B = x^2$. curve. Now we determine the intersections of these curves. The first intersection is at $(0, 0)$, the second intersection is at $(1, 1)$. So the figure lies between the lines $x = 0$ and $x = 1$. The area content is therefore equal

$$\int_0^1 [(2x - x^2) - x^2] dx = 2\int_0^1 (x - x^2) dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}.$$

If we are to calculate the contents of the shape bounded by the curves $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ and $f(x) \leq g(x)$ are both $f(x) \leq g(x)$, for $x \in \langle a, b \rangle$, then the shape can be divided into parts $S_1, S_2, \ldots, S_n$ and their contents $A_1, A_2, \ldots, A_n$ can be calculated as shown in the following image. The resulting content of the figure is then equal to the sum of all parts, i.e

$$A = A_1 + A_2 + \ldots + A_n.$$

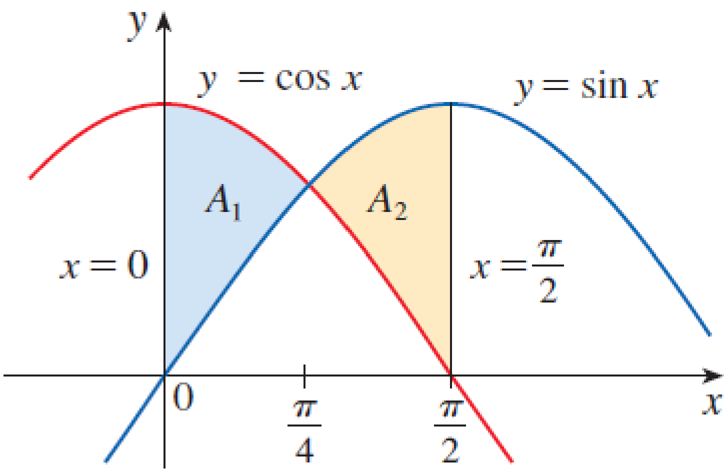


Theorem. The content of the figure between the curves $y = f(x)$ and $y = g(x)$ where $x \in \langle a, b\rangle$ is equal to

$$A = \int_a^b |f(x) - g(x)|\, dx.$$

Example. Let's find the content of the figure between the curves $y = \sin x$ and $y = \cos x$ on the interval $x \in \langle 0, \pi/2\rangle$.

We determine the points where the curves intersect when we put $\sin x = \cos x$. So $x = \pi/4$ on the $x \in \langle 0, \pi/2\rangle$ interval. See image:



Note that there is $\cos x \geq \sin x$ on the interval $\langle 0, \pi/4\rangle$ and $\sin x \geq \cos x$ is on the interval $\langle \pi/4, \pi/2\rangle$. I mean

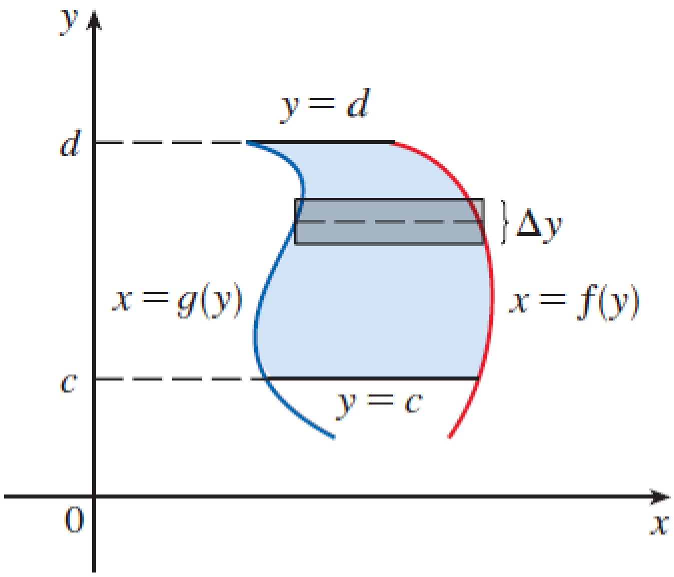
$$
\begin{aligned}
A &= \int_0^{\pi/2} |\cos x - \sin x|\, dx = A_1 + A_2 \\
&= \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx \\
&= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2} \\
&= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) + \left(-0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) \\
&= 2\sqrt{2} - 2.
\end{aligned}
$$

Note: The above procedure can be significantly simplified if symmetry is taken into account:

$$A = 2A_1 = 2\int_0^{\pi/4} (\cos x - \sin x) dx.$$

Content between two curves: integration with respect to y

Consider the figure bounded by the curves $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$, where $f(y)$ and $g(y)$ are continuous functions and $f(y) \geq g(y)$ for $c \leq y \leq d$, See figure.

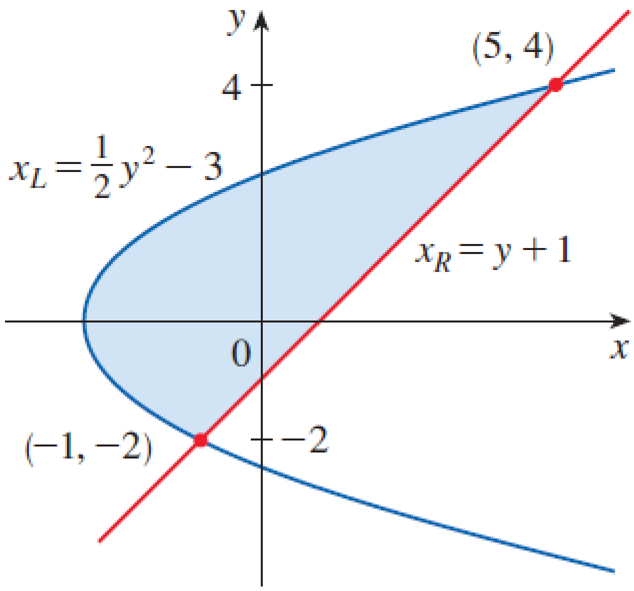


Now the content of the A figure can be expressed as an integral with respect to the integration variable y:

$$A = \int_c^d [f(y) - g(y)]\, dy.$$

Example. Let's calculate the content of the shape A bounded by the curves $y = x - 1$ and $y^2 = 2x + 6$.

Solution. The curves $y = x - 1$ and $y^2 = 2x + 6$ intersect at the points $(-1, -2)$ and $(5, 4)$, i.e. $x_L = g(y) = \frac{1}{2}y^2 - 3$ and $x_R = f(y) = y + 1$. See picture.



The content of the shape A bounded by the curves $y = x - 1$ and $y^2 = 2x + 6$ is equal to 18 units squared, as calculated by integration with respect to the variable y. Now let's summarize the procedure:

1. We express x from both equations:

a) $x = y + 1$ b) $x = \frac{1}{2}(y^2 - 6)$

2. We determine the integration interval according to the y values. The intersections are $P_1(-1, -2)$ and $P_2(5, 4)$. The integration interval is from -2 to 4.

3. We construct and calculate the integral:

$$A = \int_{-2}^{4} \left[(y + 1) - \frac{1}{2}(y^2 - 6) \right] dy$$

$$A = \int_{-2}^{4} (y + 1) dy - \int_{-2}^{4} \frac{1}{2}(y^2 - 6) dy$$

$$\begin{aligned} \int_{-2}^{4} (y + 1) dy &= \left[\frac{y^2}{2} + y \right]_{-2}^{4} \\ &= \left[\frac{16}{2} + 4 \right] - \left[\frac{4}{2} - 2 \right] \\ &= 8 + 4 - (2 - 2) \\ &= 12 \end{aligned}$$

$$\int_{-2}^{4} \frac{1}{2}(y^2 - 6)dy = \frac{1}{2}\left[\frac{y^3}{3} - 6y\right]_{-2}^{4}$$

$$= \frac{1}{2}\left[\frac{64}{3} - 24\right] - \frac{1}{2}\left[\frac{-8}{3} + 12\right]$$

$$= \frac{1}{2}\left[\frac{64 - 72}{3}\right] - \frac{1}{2}\left[\frac{-8 + 36}{3}\right]$$

$$= \frac{1}{2}\left[\frac{-8}{3}\right] - \frac{1}{2}\left[\frac{28}{3}\right]$$

$$= \frac{-4}{3} - \frac{14}{3}$$

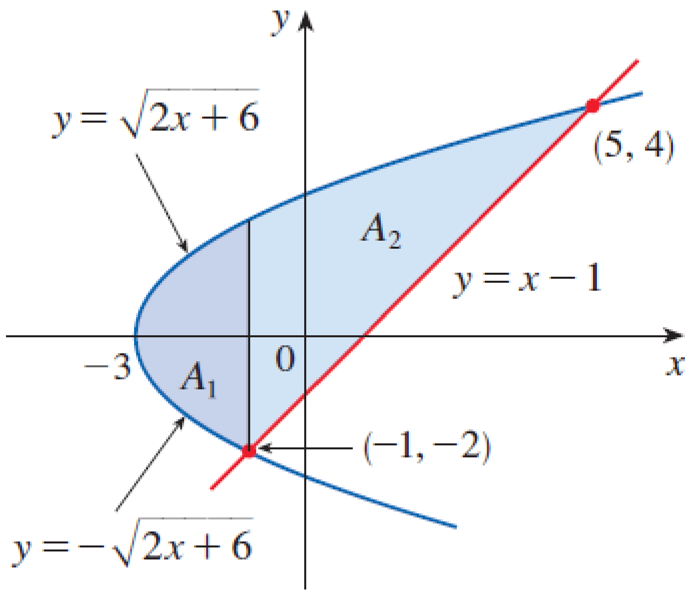
$$= \frac{-18}{3}$$

$$= -6$$

$A = 12 - (-6) = 12 + 6 = 18$

So the content of the shape A bounded by the curves $y = x - 1$ and $y^2 = 2x + 6$ is 18.

Note. The content of the shape A bounded by the curves $y = x - 1$ and $y^2 = 2x + 6$ can also be determined using integration with respect to the variable x. In this case, the calculation is more complicated. The lower boundary curve consists of two curves: $y = -\sqrt{2x + 6}$ and $y = x - 1$. The upper limit curve is $y = \sqrt{2x + 6}$. We then divide the form into two parts whose contents we label A_1 and A_2. See the following image:



We calculate the contents of A_1 and A_2:

a) A_1 is bounded above by the curve $y = \sqrt{2x + 6}$ and below by the curve $y = -\sqrt{2x + 6}$.

$A_1 = \int_{-3}^{-1}(\sqrt{2x+6} - (-\sqrt{2x+6}))dx = \int_{-3}^{-1}(2\sqrt{2x+6})dx$

b) A_2 is bounded above by the curve $y = \sqrt{2x+6}$ and below by the curve $y = x - 1$.

$A_2 = \int_{-1}^{5}(\sqrt{2x+6} - (x-1))dx$

We calculate the integrals and add the contents of A_1 and A_2 for the total contents of the form A.

In [ ]:
```
# Určíme hodnotu A_1
import sympy as sp

# definujme symboly x a y
x, y = sp.symbols('x y')
expr = 2 * sp.sqrt(2 * x + 6)  # výraz pro výpočet A_1
a, b = -3, -1

A1 = sp.integrate(expr, (x, a, b))

print("A_1 = ", A1)

# Určíme hodnotu A_2
expr2 = sp.sqrt(2 * x + 6) - (x - 1)   # výraz pro výpočet A_2
a, b = -1, 5

A2 = sp.integrate(expr2, (x, a, b))
print("A_2 = ", A2)

print("Obsah pomocí integrace vzhledem k x")
print("A_1 + A_2 = ", A1 + A2)

# Určíme hodnotu A pomocí integrace vzhledem k y
expr3 = (y + 1) - (1/2) * y**2 + 3   # výraz pro výpočet A
a, b = -2, 4

A = sp.integrate(expr3, (y, a, b))
print()
print("Obsah pomocí integrace vzhledem k y")
print("A = ", A)
```

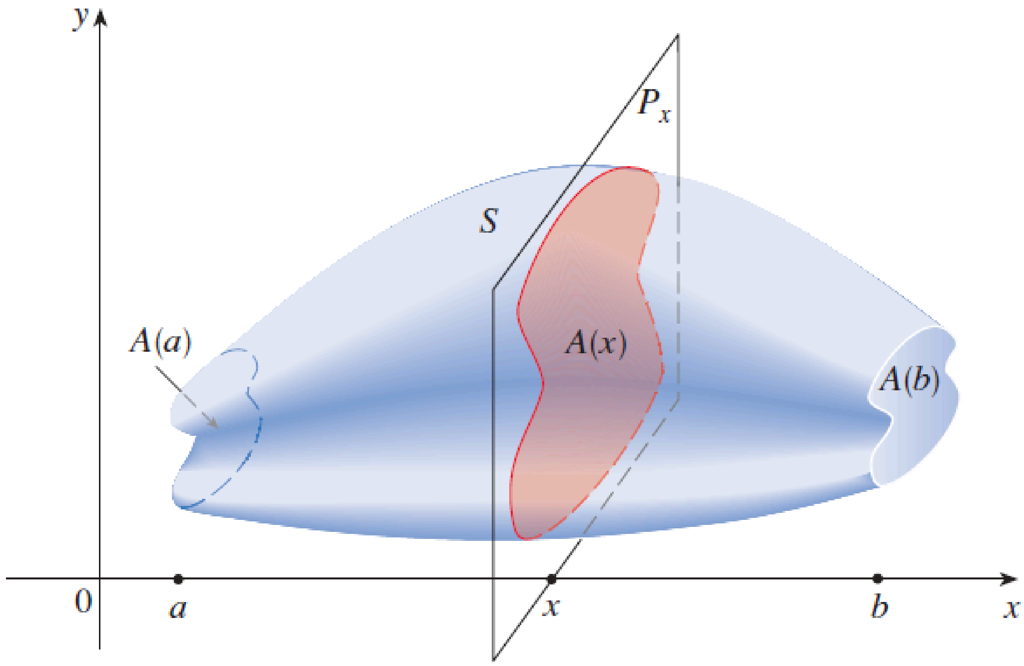
```
A_1 =  16/3
A_2 =  38/3
Obsah pomocí integrace vzhledem k x
A_1 + A_2 =  18

Obsah pomocí integrace vzhledem k y
A =  18.0000000000000
```

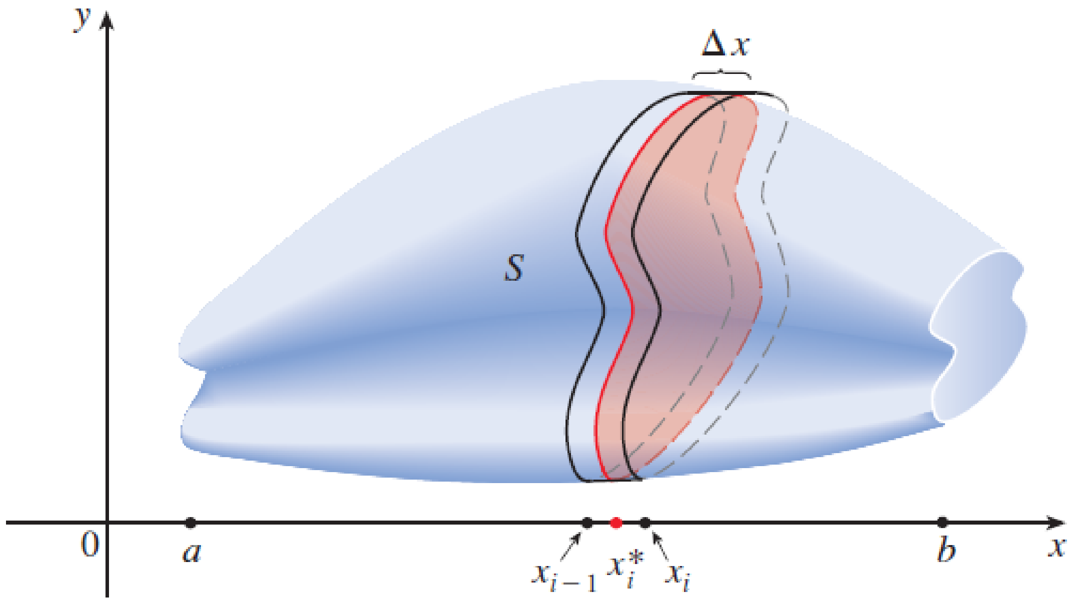
Volume of the rotating body

Definition of volume

Definition. Let S be the solid lying between the planes $x = a$ and $x = b$. Furthermore, let $A(x)$ denote the content of the section S by the plane P_x, where P_x is the plane perpendicular to the axis x and passing through the point x. If for $x \in \langle a, b\rangle$ $A(x)$ depends continuously on x, then the volume S is denoted as

$$V(S) = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx.$$



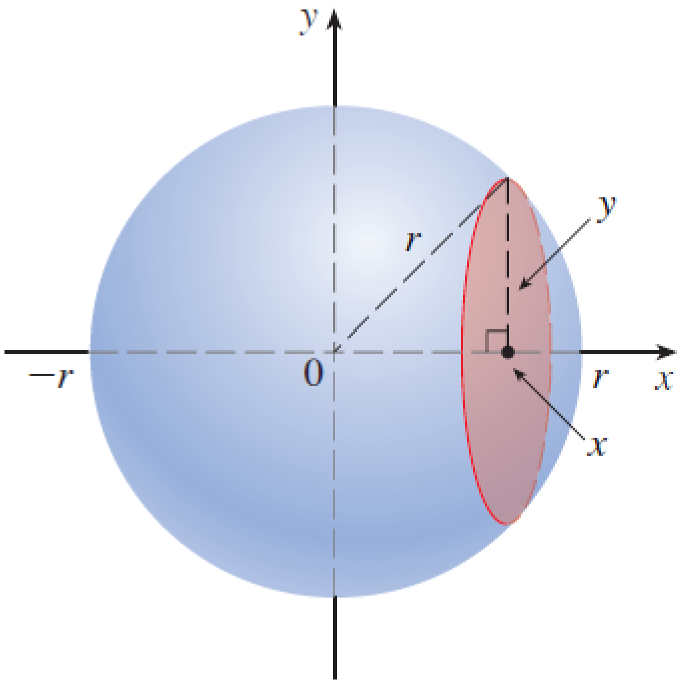


Example. Let's show that it holds for the volume of a sphere of radius r:

$$V = \frac{4}{3}\pi r^3$$

Solution. Here, $a = -r$, $b = r$,

$$A(x) = \pi y^2 = \pi(r^2 - x^2), \quad \text{kde } y = \sqrt{r^2 - x^2}.$$

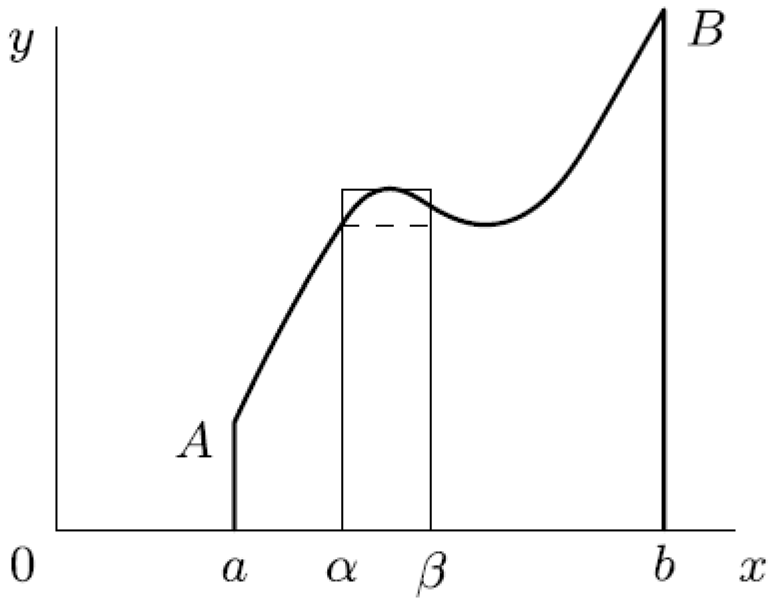


$A(x)$ is a continuous function for $x \in \langle -r, r \rangle$. Using the definition of volume we get:

$$\begin{aligned} V &= \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx \\ &= 2\pi \int_0^r (r^2 - x^2)\,dx \\ &= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r \\ &= 2\pi \left[r^3 - \frac{r^3}{3} \right] \\ &= \frac{4}{3}\pi r^3. \end{aligned}$$

Formula for the volume of a rotating body

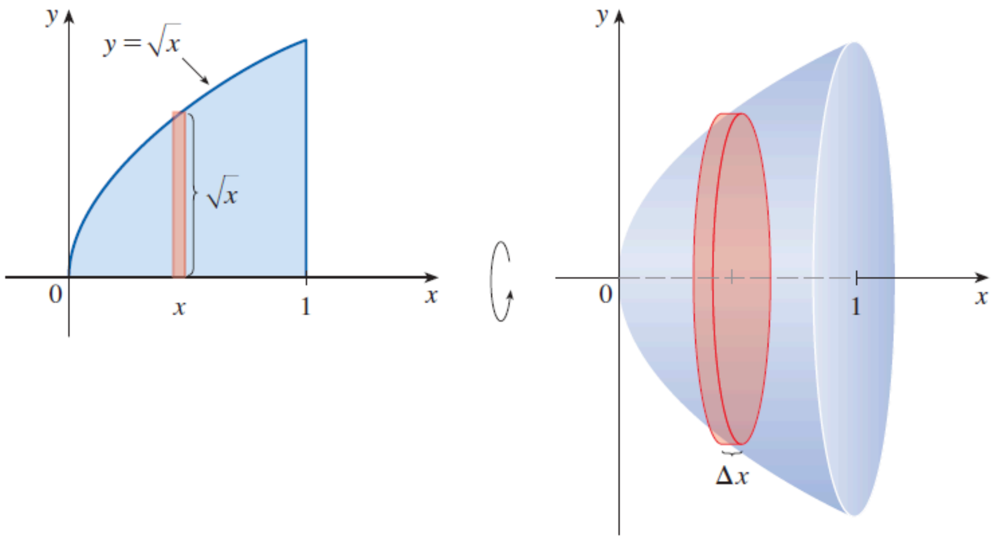
Now consider that the figure bounded by the graph of the positive continuous function $y = f(x)$ and $x \in \langle a, b \rangle$. rotates around the axis x See figure.



Then for the volume:

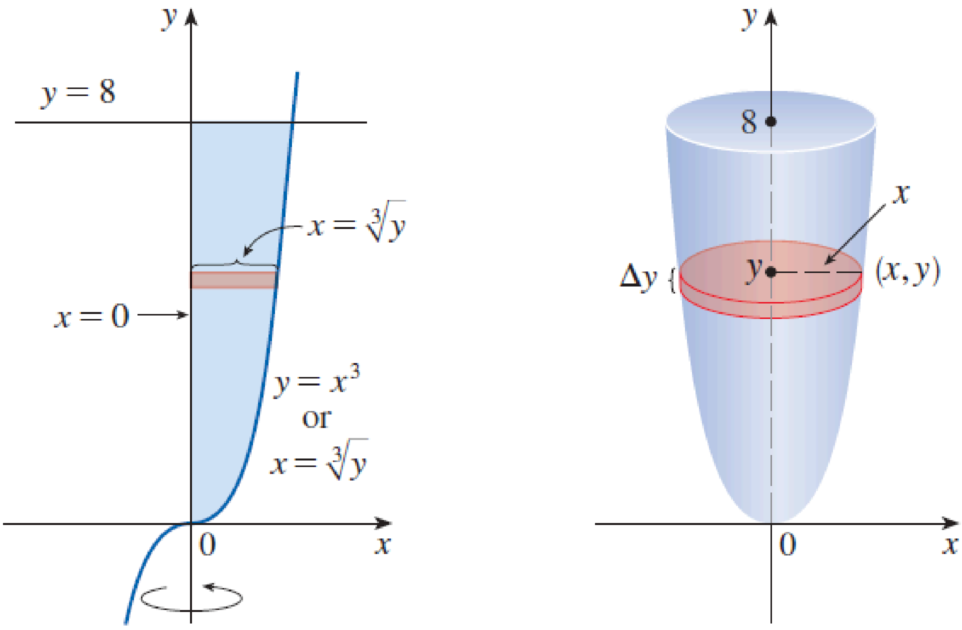
$$V = \pi\int_a^b [f(x)]^2 dx.$$

Example 1. Calculate the volume of the solid that is created by rotating the surface figure under the curve $y = \sqrt{x}$ around the axis x and where $x \in \langle 0, 1\rangle$. See image below.



$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x\,dx = \pi\left[\frac{x^2}{2}\right]_0^1 = \frac{\pi}{2}.$$

Example. Calculate the volume of the solid formed by rotating the figure bounded by the curve $y = x^3$, $y = 8$ and $x = 0$ around the y axis. See image:



Solution. $x = \sqrt[3]{y}$ a applies here

$$A(y) = \pi(x)^2 = \pi(\sqrt[3]{y})^2 = \pi y^{2/3}.$$

$$A(y)\Delta y = \pi y^{2/3}\Delta y.$$

Now valid for volume V:

$$V = \int_0^8 A(y)dy = \pi\int_0^8 y^{2/3}dy = \pi\left[\frac{3}{5}y^{5/3}\right]_0^8 = \pi\frac{96}{5}.$$

□

Other integration techniques

Intergation by the per partes method

Integration by the method per partes: indefinite integral

Assume that the functions $f(x)$ and $g(x)$ have continuous derivatives on the interval (a, b), then the formula holds:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx.$$

Proof. By assumption, there exists on the interval (a, b) a primitive function $G(x)$ to the function $f'(x)g(x)$. Then, if we put on the interval (a, b): $F(x) = f(x)g(x) - G(x)$, then for every $x \in (a, b)$:

$$F'(x) = (f(x)g(x) - G(x))' = (f(x)g(x))' - G'(x) = f'(x)g(x) + f(x)g'(x) - G'(x) = f'(x)g(x) + f(x)g'(x$$

Thus, the function $F(x)$ is a primitive function to $f(x)g'(x)$. □

The formula for the derivative of the product:

$$(u(x) \cdot v(x))' = u'(x) \cdot v(x) + u(x) \cdot v'(x).$$

Example. Let's find the indefinite integral $\int x\sin x dx$ by the per partes method.

Solution. $f(x) = x$ and $g'(x) = \sin x$ apply here. Then according to the formula:

$$\int x\sin x dx = f(x)g(x) - \int f'(x)g(x)dx = x(-\cos x) - \int(-\cos x)dx =$$
$$= x(-\cos x) + \sin x = -x\cos x + \sin x + C.$$

Example. Let's find the indefinite integral $\int \ln x\, dx$ by the per partes method.

Solution. $f(x) = \ln x$ and $g'(x) = 1$ apply here. Then according to the formula:

$$\int \ln x\, dx = f(x)g(x) - \int f'(x)g(x)dx = x\ln x - \int(\frac{1}{x}) \cdot xdx =$$
$$= x\ln x - x + C.$$

Integration by the method per partes: definite integral

Given functions $f(x)$ and $g(x)$ which have continuous derivatives. Then the formula applies:

$$\int_a^b f(x)g'(x)dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx.$$

Proof. The statement follows from the fact that the theorem on integration by the per partes method holds for the indefinite integral and from the fundamental theorem of the integral calculus. □

Example. Let's find a definite integral $\int_0^1 \arctan x\, dx$ by the per partes method.

Solution. $f(x) = \arctan x$ and $g'(x) = 1$ apply here. Then according to the formula:

$$\int_0^1 \arctan x\, dx = [f(x)g(x)]_0^1 - \int_0^1 f'(x)g(x)dx =$$
$$= [x\arctan x]_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx =$$
$$= \arctan 1 - \int_0^1 \frac{x}{1+x^2}\, dx =$$
$$= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}\, dx.$$

We calculate the last integral using the substitution method: $u = x^2$ and then $du = 2xdx$. Then:

$$(1/2)\int_0^1 \frac{2x}{1+x^2}\,dx =$$

$$= \frac{1}{2}\int_0^1 \frac{1}{1+u}\,du = \frac{1}{2}\ln(1+u)\Big|_0^1 = \frac{1}{2}\ln 2.$$

$$\int_0^1 \arctan(x)dx = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

Integration of rational linear functions

A rational inverse function is a function of the form:

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials.

Recall that the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$ where $a_n \neq 0.$

A rational inverse function $f(x)$ is **pure** if $st(Q(x)) > st(P(x))$. In the case where $st(Q(x)) \leq st(P(x))$, then after division we get:

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where $S(x)$ and $R(x)$ are polynomials and $st(R(x)) < st(Q(x))$.

Example. Let's find the indefinite integral $\int \frac{x^3+x}{x-1}\,dx$.

Solution. $P(x) = x^3 + x$ and $Q(x) = x - 1$ apply here. Then the function $f(x) = \frac{P(x)}{Q(x)}$ is not pure because $st(Q(x)) = 1 \leq st(P(x)) = 3$. Then after division we get:

$$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)} = x^2 + x + 2 + \frac{2}{x-1}.$$

In [ ]:
```
# Vydělme P(x) / Q(x)
import sympy as sp
from IPython.display import display, Math, Latex
x = sp.Symbol('x')
P = x**3 + x
Q = x - 1
# vydělme P(x) / Q(x)
R = P / Q
# najděme podíl a zbytek
R1 = sp.div(P, Q)
# vypíšeme podíl a zbytek
display(Math(r"%s = %s + \frac{%s}{%s}" % (sp.latex(R), sp.latex(R1[0]), sp.late
```

$$\frac{x^3+x}{x-1} = x^2+x+2+\frac{2}{x-1}$$

Now we can integrate:

$$\int \frac{x^3+x}{x-1}\,dx = \int x^2\,dx + \int x\,dx + \int 2\,dx + \int \frac{2}{x-1}\,dx =$$
$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C.$$

Let us further assume that the $Q(x)$ polynomial can be decomposed into a product of linear terms:

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\ldots(a_kx+b_k).$$

In this case, one can find $A_1, A_2, \ldots, A_k$ constants such that:

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \cdots + \frac{A_k}{a_kx+b_k}.$$

Example. Let's find the indefinite integral $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$.

Solution. Since the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator, this is a pure inverse function. Furthermore, the $Q(x)$ polynomial can be decomposed into a product of linear terms:

$$Q(x) = 2x^3+3x^2-2x = x(2x^2+3x-2) = x(2x-1)(x+2).$$

Thus:

$$\frac{R(x)}{Q(x)} = \frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}.$$

If we convert the fractions on the right side to a common denominator, we get by comparing the left and right sides:

$$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1).$$

After multiplying the brackets on the right-hand side and comparing the coefficients in the respective powers of x, we get:

$$\begin{aligned} 2A+B+2C &= 1, \\ 3A+2B-C &= 2, \\ -2A &= -1. \end{aligned}$$

Solving the system gives $A = \frac{1}{2}, B = \frac{1}{5}, C = -\frac{1}{10}$. Then:

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \int \frac{x^2+2x-1}{x(2x-1)(x+2)}\,dx =$$
$$= \int \left(\frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}\right) dx =$$
$$= \int \left(\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2}\right) dx =$$
$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C.$$

Suppose that the factor of type $(ax+b)^r$, where $r > 1$, is contained in the factorization of the polynomial $Q(x)$. In this case, the decomposition of the $f(x)$ rational fractional function will contain partial fractions:

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_r}{(ax+b)^r}.$$

For example:

$$\frac{x^3-x+1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}.$$

Example. Let's find the indefinite integral $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$.

Solution. $P(x) = x^4 - 2x^2 + 4x + 1$ and $Q(x) = x^3 - x^2 - x + 1$ apply here. Then the function $f(x) = \frac{P(x)}{Q(x)}$ is not pure because $st(Q(x)) = 3 < st(P(x)) = 4$. Then after division we get:

$$\frac{P(x)}{Q(x)} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}.$$

Let's decompose the $Q(x)$ polynomial into a product of linear terms:

$$Q(x) = x^3 - x^2 - x + 1 = (x-1)^2(x+1).$$

Then we have:

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}.$$

Converting to a common denominator, we get the equality:

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$
$$= (A+C)x^2 + (B-2C)x + (-A+B+C).$$

By comparing the coefficients of the respective x powers, we get:

$$\begin{aligned} A + C &= 0, \\ B - 2C &= 4, \\ -A + B + C &= 0. \end{aligned}$$

By solving the system, we get $A = 1, B = 2, C = -1$. Then:

$$\begin{aligned} \int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx &= \int \left[x + 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} - \frac{1}{x+1} \right] dx = \\ &= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C \\ &= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + C. \end{aligned}$$

Now suppose that the polynomial $Q(x)$ has in its factorization a factor of type $ax^2 + bx + c$, where $b^2 - 4ac < 0$, $a \neq 0$. In this case, the decomposition of the rational inverse function $f(x)$ will contain a partial fraction of the form:

$$\frac{Ax + B}{ax^2 + bx + c}.$$

For example, if we have a rational linear function $f(x) = \frac{x}{(x-2)(x^2+1)(x^2+4)}$, then its decomposition will be:

$$\frac{x}{(x-2)(x^2+1)(x^2+4)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+1} + \frac{Dx + E}{x^2+4}.$$

To find the indefinite integral $\int \frac{Ax+B}{ax^2+bx+c}\,dx$ we use the formula:

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C.$$

Example. Let's find the indefinite integral $\int \frac{2x^2-x+4}{x^3+4x}\,dx$.

Solution. $x^3 + 4x = x(x^2 + 4)$ applies here. Then we have:

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}.$$

If we now multiply the given equality by the expression $x(x^2 + 4)$, we get:

$$\begin{aligned} 2x^2 - x + 4 &= A(x^2 + 4) + (Bx + C)x \\ &= (A + B)x^2 + Cx + 4A. \end{aligned}$$

By comparing the coefficients of the respective x powers, we get:

$$A + B = 2, \quad C = -1, \quad 4A = 4.$$

Then:

$$A = 1, \quad B = 1, \quad C = -1.$$

From here we have:

$$\int \frac{2x^2 - x + 4}{x(x^2 + 4)}\, dx = \int \left(\frac{1}{x} + \frac{x - 1}{x^2 + 4} \right) dx = \int \frac{dx}{x} + \int \frac{x - 1}{x^2 + 4}\, dx.$$

First, let's look for the second of the integrals:

We decompose the integrand in the following way:

$$\frac{x - 1}{x^2 + 4} = \frac{x}{x^2 + 4} - \frac{1}{x^2 + 4}.$$

So let's integrate:

$$\int \frac{x - 1}{x^2 + 4}\, dx = \int \frac{x}{x^2 + 4}\, dx - \int \frac{1}{x^2 + 4}\, dx.$$

The first of the integrals on the right-hand side can be simplified using the $u = x^2 + 4$ substitution:

$$\int \frac{x}{x^2 + 4}\, dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2 + 4| + C = \frac{1}{2} \ln(x^2 + 4) + C.$$

The second of the integrals on the right-hand side can be found according to the above formula:

$$\int \frac{1}{x^2 + 4}\, dx = \frac{1}{2} \arctan\left(\frac{x}{2} \right) + C.$$

Finally, we can calculate the indefinite integral $\int \frac{2x^2 - x + 4}{x(x^2 + 4)}\, dx$:

$$\int \frac{2x^2 - x + 4}{x(x^2 + 4)}\, dx = \int \frac{dx}{x} + \int \frac{x - 1}{x^2 + 4}\, dx =$$

$$= \ln |x| + \frac{1}{2} \ln(x^2 + 4) + \frac{1}{2} \arctan\left(\frac{x}{2} \right) + C.$$

end 5/3/2024

Here, assume that the polynomial $Q(x)$ has a factor of type $(ax^2 + bx + c)^r$ in its decomposition, where $b^2 - 4ac < 0$. In this case, the decomposition of the rational cursive function $f(x)$ will contain partial fractions of the form:

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_rx+B_r}{(ax^2+bx+c)^r}.$$

Example. Let's find the decomposition into partial fractions of a rational linear function

$$f(x)=\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}.$$

Solution. Here:

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}=\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+x+1}+\frac{Ex+F}{x^2+1}+\frac{Gx+H}{(x^2+1)^2}+\frac{Ix+J}{(x^2+1)^3}.$$

Example. Let's find the indefinite integral

$$\int\frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx.$$

Solution. Here:

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}.$$

If we multiply this equality by the expression $x(x^2+1)^2$, we get:

$$\begin{aligned}1-x+2x^2-x^3&=A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x\\&=A(x^4+2x^2+1)+B(x^4+x^2)+C(x^3+x)+Dx^2+Ex\\&=(A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A.\end{aligned}$$

It follows that:

$$A+B=0,\quad C=-1,\quad 2A+B+D=2,\quad C+E=-1,\quad A=1.$$

So the solution is: $A=1, B=-1, C=-1, D=1, E=0$. Then the given indefinite integral can be put in the form:

$$\begin{aligned}\int\frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx&=\int\left(\frac{1}{x}-\frac{x+1}{x^2+1}+\frac{x}{(x^2+1)^2}\right)dx=\\&=\int\frac{1}{x}\,dx-\int\frac{x}{x^2+1}\,dx-\int\frac{dx}{x^2+1}+\int\frac{xdx}{(x^2+1)^2}=\\&=\ln|x|-\frac{1}{2}\ln(x^2+1)-\arctan x-\frac{1}{2(x^2+1)}+C.\end{aligned}$$

Example. Let's put

$$I_k = \int \frac{dx}{(x^2+a^2)^k}, \quad k = 1, 2, 3, \ldots.$$

Integrating per partes we find that:

$$\begin{aligned} I_k = \int \frac{dx}{(x^2+a^2)^k} &= \frac{x}{(x^2+a^2)^k} + 2k \int \frac{x^2 dx}{(x^2+a^2)^{k+1}} = \\ &= \frac{x}{(x^2+a^2)^k} + 2k \int \frac{(x^2+a^2) - a^2}{(x^2+a^2)^{k+1}} \, dx = \\ &= \frac{x}{(x^2+a^2)^k} + 2kI_k - 2ka^2 I_{k+1}. \end{aligned}$$

From here the recursive formula can be derived:

$$I_{k+1} = \frac{1}{2ka^2} \frac{x}{(x^2+a^2)^k} + \frac{2k-1}{2ka^2} I_k.$$

Recall that:

$$I_1 = \int \frac{dx}{x^2+a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C.$$

In [ ]:
```
# hledejme neurčitý integrál funkce f(x)
import sympy as sp
from sympy import *
from IPython.display import display, Math, Latex
x = sp.Symbol('x')
f = (4 * x**2 - 6 * x + 5)/ (x**2 - 4 * x + 5)
int_f = sp.integrate(f, x)
display(Math(r"\int %s \, dx = %s + C" % (sp.latex(f), sp.latex(int_f))))

derivace = sp.diff(4 * x + 5 * sp.log(x**2 - 4 * x + 5) + 5 * sp.atan(x - 2), x)
display(Math(r"\frac{d}{dx} (%s + C)  = %s" % (sp.latex(int_f), sp.latex(derivac
```

$$\int \frac{4x^2 - 6x + 5}{x^2 - 4x + 5} \, dx = 4x + 5\log\left(x^2 - 4x + 5\right) + 5\operatorname{atan}(x-2) + C$$

$$\frac{d}{dx}(4x + 5\log\left(x^2 - 4x + 5\right) + 5\operatorname{atan}(x-2) + C) = \frac{4x^2 - 6x + 5}{x^2 - 4x + 5}$$

$$x^2 + x + 2 = (x + \frac{1}{2})^2 + \frac{7}{4}.$$

If we put

$$u = x + \frac{1}{2},$$

Well then

$$(x+\frac{1}{2})^2+\frac{7}{4}=u^2+\frac{7}{4}.$$

For example, let $Q(x)=2x^2+3x+5$. Transform this trinomial into the form $Q(x)=2\cdot(u^2+a^2)$.

$$Q(x)=2x^2+3x+5=2(x^2+\frac{3}{2}x+\frac{5}{2})=2(x^2+\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}+\frac{5}{2})=$$

$$=2[(x+\frac{3}{4})^2+\frac{31}{16}].$$

Let's then calculate the integral $$ \int \frac{dx}{Q(x)} = \frac{1}{2}\int \frac{dx}{(x + \frac{3}{4})^2 + \frac{31}{16}} =

= \left |u = x + \frac{3}{4}\right | = \frac{1}{2}\int\frac{du}{u^2 + \frac{31}{16}} = $$

end of 12.3.

Improper integrals

Improper integrals due to the improper limit

Definition. (Improper integral due to improper limit).

(a) If $\int_a^t f(x)\,dx$ exists where $t \geq a$ then we define:

$$\int_a^\infty f(x)\,dx=\lim_{t\to\infty}\int_a^t f(x)\,dx,$$

if there is a limit on the right-hand side and it is finite.

(b) If $\int_t^b f(x)\,dx$ exists where $t \leq b$ then we define:

$$\int_{-\infty}^b f(x)\,dx=\lim_{t\to-\infty}\int_t^b f(x)\,dx,$$

if there is a limit on the right-hand side and it is finite.

We say the $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ improper integrals **converge** if the corresponding limits exist and are finite. Furthermore, we say they **diverge** if the corresponding limits do not exist or are nonproprietary.

(c) If both stepintergals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ converge, then we define:

$$\int_{-\infty}^\infty f(x)\,dx=\int_a^\infty f(x)\,dx+\int_{-\infty}^a f(x)\,dx.$$

On the right side, you can choose $a \in \mathrm{R}$ arbitrarily.

Example. Let's calculate the value of the improper integral:

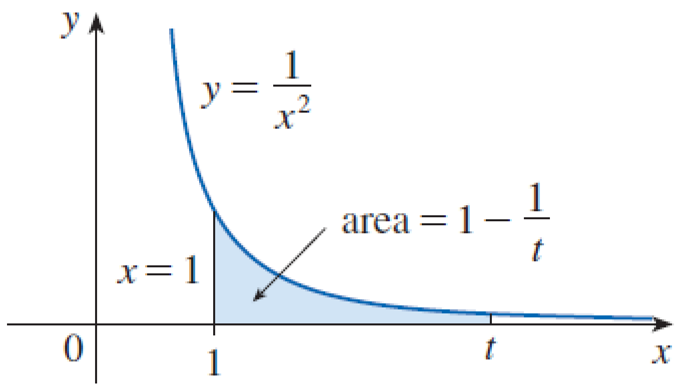
$$\int_1^{\infty} \frac{1}{x^2}\, dx.$$

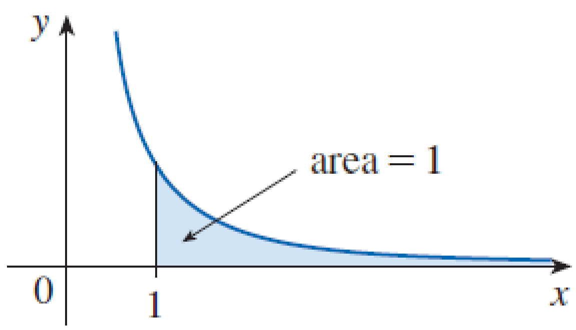
Solution. We find that:

$$A(t) = \int_1^{t} \frac{1}{x^2}\, dx = \left[-\frac{1}{x} \right]_1^t = 1 - \frac{1}{t}.$$

Then you can write:

$$\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{t \to \infty} A(t) = \lim_{t \to \infty} \left(1 - \frac{1}{t} \right) = 1.$$





Example. Let's see if the integral $\int_1^{\infty} \frac{1}{x} \ln x\, dx$ converges.

By part (a) of the definition, we have:

$$\int_1^{\infty} \frac{1}{x}\, dx = \lim_{t \to \infty} \int_1^{t} \frac{1}{x}\, dx = \lim_{t \to \infty} [\ln t - \ln 1] = \lim_{t \to \infty} \ln t = \infty.$$

The given limit does not exist as a final limit and thus the integral $\int_1^{\infty} \frac{1}{x}\, dx$ diverges.

Example. Let's calculate the $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$ integral.

Solution. By part (c) of the definition, we have:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx.$$

Next, let's calculate both step integrals on the right-hand side:

$$\begin{aligned}\int_{0}^{\infty} \frac{1}{x^2+1}\,dx &= \lim_{t\to\infty} \int_{0}^{t} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} [\arctan x]_0^t \\ &= \lim_{t\to\infty} [\arctan t - \arctan 0] = \lim_{t\to\infty} \arctan t = \frac{\pi}{2}.\end{aligned}$$

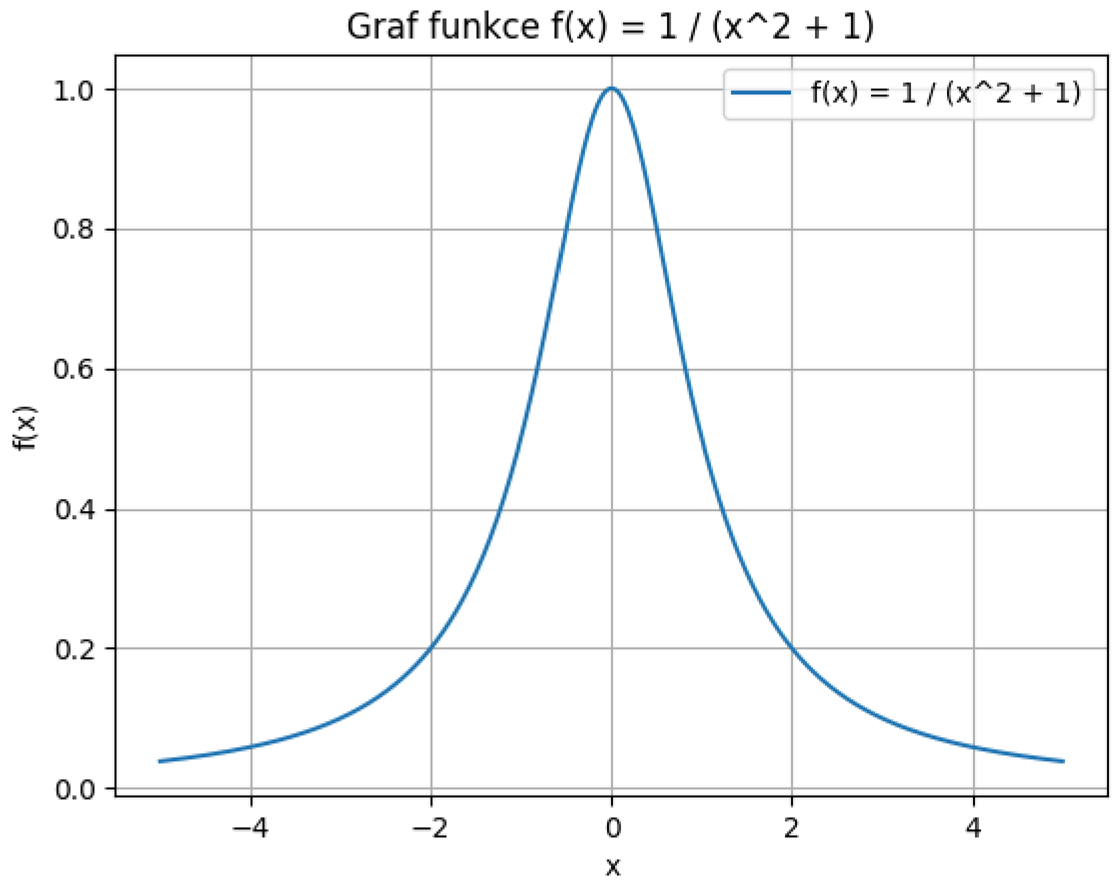
$$\begin{aligned}\int_{-\infty}^{0} \frac{1}{x^2+1}\,dx &= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{x^2+1}\,dx = \lim_{t\to-\infty} [\arctan x]_t^0 \\ &= \lim_{t\to-\infty} [\arctan 0 - \arctan t] = \lim_{t\to-\infty} (-\arctan t) = \frac{\pi}{2}.\end{aligned}$$

I mean

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

In [ ]:
```python
# @title
# Zobrazme graf funkce 1/(x^2 + 1)
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math, Latex

# Initialize symbolic variable and define the function
x = sp.Symbol('x')
f = 1 / (x**2 + 1)

# Display the function using LaTeX representation
display(Math(r"f(x) = %s" % sp.latex(f)))

# Create an array of x-values and compute the corresponding y-values
x_vals = np.linspace(-5, 5, 400)
f_lambdified = sp.lambdify(x, f, "numpy")
y_vals = f_lambdified(x_vals)

# Plot the function using matplotlib
plt.plot(x_vals, y_vals, label='f(x) = 1 / (x^2 + 1)')
plt.title('Graf funkce f(x) = 1 / (x^2 + 1)')
plt.xlabel('x')
plt.ylabel('f(x)')
plt.grid(True)
plt.legend()
plt.show()
```

$f(x) = \dfrac{1}{x^2 + 1}$



Example. Let's calculate that the function $\varphi : \mathrm{R} \to \mathrm{R}$ is an even function, i.e.

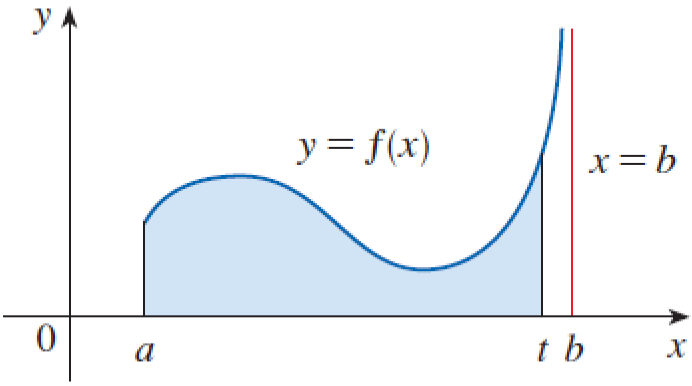
$$\varphi(-x) = \varphi(x) \quad \text{pro všechna } x \in \mathrm{R}.$$

Further suppose that the improper integral $\int_x^\infty \varphi(s)\, ds$ for some $x \in \mathrm{R}$ converges. After that $$

Improper integrals due to the unlimitedness of the integrand

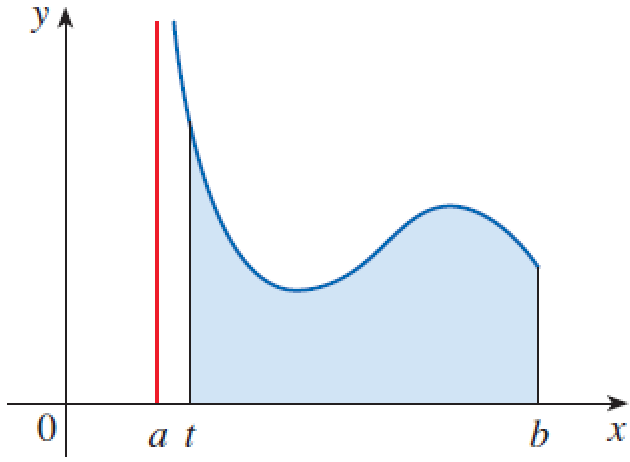
Definition. (a) Let the function f have the integral $\int_a^t f(x)\, dx$ for every $t \in \langle a, b)$ and let the function f have a vertical asymptote for $x = b$ (See figure.) If there is a finite limit $\lim_{t \to b-} \int_a^t f(x)\, dx$, then we define:

$$\int_a^b f(x)\, dx = \lim_{t \to b-} \int_a^t f(x)\, dx.$$



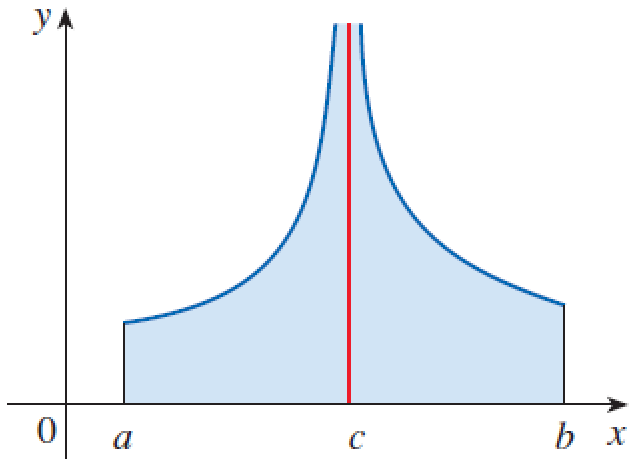
(b) Let the function f have the integral $\int_t^b f(x)\,dx$ for every $t \in (a, b\rangle$ and let the function f have a vertical asymptote for $x = a$ (See figure.) If there is a finite limit $\lim_{t\to a+}\int_t^b f(x)\,dx$, then we define:

$$\int_a^b f(x)\,dx = \lim_{t\to a+}\int_t^b f(x)\,dx.$$



(c) Let $a < c < b$ and let the integrals $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ be convergent. Then we define:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$



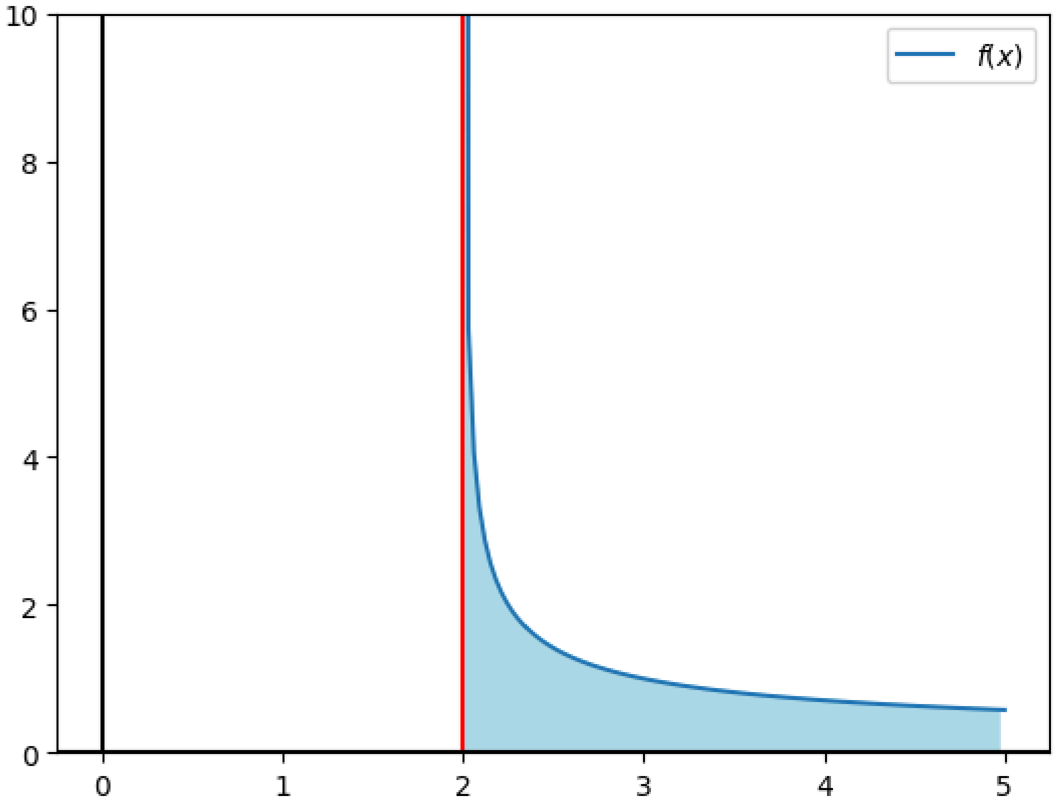
Example: Let's investigate the convergence of the improper integral $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$.

Solution. Note that the function $f(x) = 1/\sqrt{x-2}$ has a vertical asymptote at the point $x = 2$. Thus, the function f is discontinuous at the leftmost point of the interval $\langle 2, 5\rangle$. We will now apply part (b) from the definition:

$$\begin{aligned}\int_2^5 \frac{dx}{\sqrt{x-2}} &= \lim_{t\to 2+} \int_t^5 \frac{dx}{\sqrt{x-2}}\\ &= \lim_{t\to 2+} \left[2\sqrt{x-2}\right]_t^5\\ &= \lim_{t\to 2+} 2(\sqrt{3} - \sqrt{t-2}) = 2\sqrt{3}.\end{aligned}$$

In [ ]:
```
# Zobrazíme graf funkce f(x) = \frac{1}{\sqrt{x - 2}}
# pomocí sympy a matplotlib
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math, Latex

display(Math(r"f(x) = \frac{1}{\sqrt{x - 2}}"))

# vytvoříme 100 hodnot x v intervalu [a,b], kde a = 2.0001 a b = 5.0
a = 2.0001
b = 5.0
x_values = np.linspace(a, b, 100)
y = 1 / np.sqrt(x_values - 2)  # použijeme numpy pro výpočet hodnot y

plt.plot(x_values, y, label=r"$f(x)$")  # r"" znamená, že řetězec je raw string
# přidejme plot přímky x = 2
plt.axvline(x=2, color='r')
# vyšrafujme plochu pod grafem na intervalu (2, 5)
plt.fill_between(x_values, y, where=(x_values > 2) & (x_values < 5), color='ligh

plt.legend()
plt.ylim(0, 10) # nastavíme rozsah osy y
# přidejme plot obou os
plt.axhline(y=0, color='k')
plt.axvline(x=0, color='k')

plt.show() # zobrazíme graf
```

$$f(x) = \frac{1}{\sqrt{x-2}}$$



Taylor polynomial

Definition. Let us have a function f on the interval (a, b) and the point $x_0 \in (a, b)$. Assume that the function f has the nth derivative $f^{(n)}(x_0)$ at the point x_0. Then the polynomial

$$P_n(x) := f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the **Taylor polynomial** of at most nth degree of the function f at the point x_0.

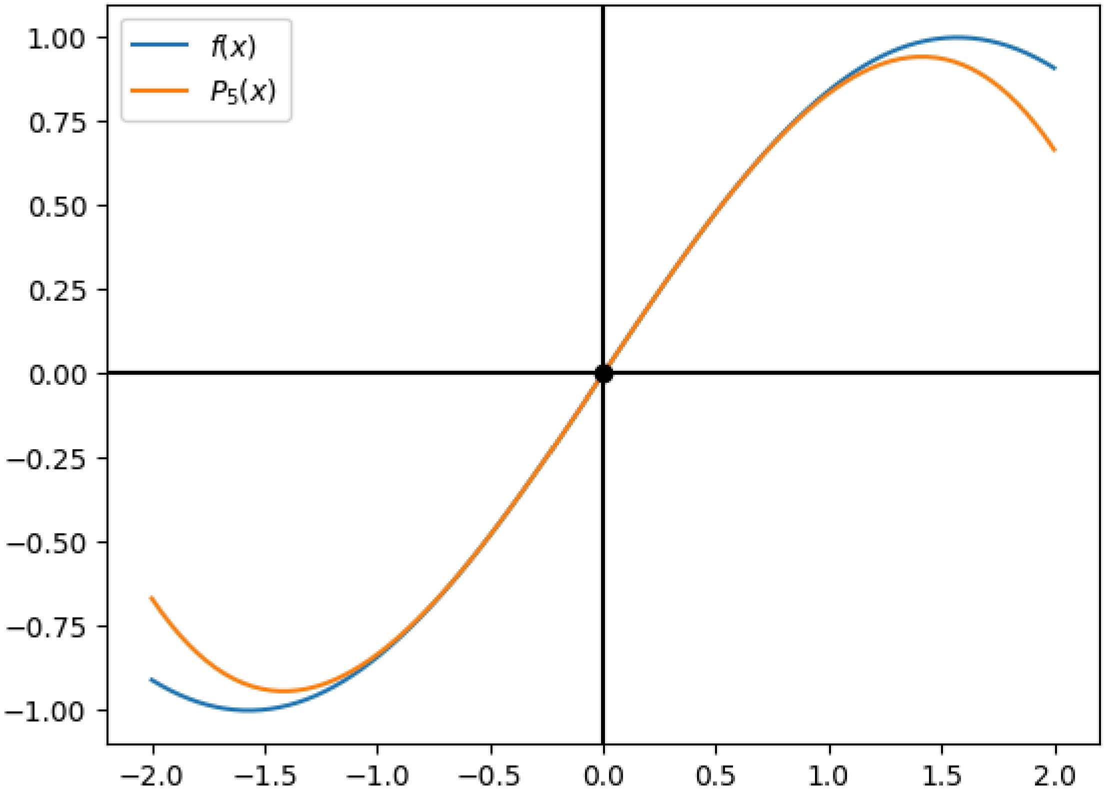
Theorem. $P_n(x_0) = f(x_0)$ and $P_n^{(k)}(x_0) = f^{(k)}(x_0)$ apply to $k = 0, 1, \ldots, n$.

In [ ]:
```
# Napišme program, který najde Taylorův polynom P_n(x) pro funkci f(x) v bodě x
# a poté vykreslí graf funkce f(x) a jejího Taylorova polynomu P_n(x) v interval
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math, Latex

sp.init_printing(use_latex='mathjax')

x = sp.Symbol('x')
f = sp.sin(x)
x0 = 0
n = 5

# Zobrazíme předpis funkce y = f(x)
display(Math(r"f(x) = %s" % sp.latex(f)))

# vytvoříme Taylorův polynom P_n(x) pro funkci f(x) v bodě x = x_0
```

```python
P = sp.series(f, x, x0, n).removeO()

# zobrazíme předpis Taylorova polynomu P_n(x)
display(Math(r"P_%s(x) = %s" % (n, sp.latex(P))))

# vytvoříme funkci P_n(x) z Taylorova polynomu P_n(x)
P_func = sp.lambdify(x, P, 'numpy')

# vytvoříme funkci f(x)
f_func = sp.lambdify(x, f, 'numpy')

# vytvoříme interval [x_0 - 1, x_0 + 1]
x_values = np.linspace(x0 - 2, x0 + 2, 100)

# vypočteme hodnoty funkce f(x) a jejího Taylorova polynomu P_n(x) v intervalu [
y1 = f_func(x_values)
y2 = P_func(x_values)

# vykreslíme graf
plt.plot(x_values, y1, label=r"$f(x)$")
plt.plot(x_values, y2, label=r"$P_%d(x)$" % n)

# zobrazíme bod dotyku  [x_0,f(x_0)]
plt.plot(x0, f_func(x0), 'o', color='black')

# zobrazíme osy x a y
plt.axhline(y=0, color='k')
plt.axvline(x=0, color='k')


plt.legend()
plt.show()
```

$$f(x) = \sin(x)$$

$$P_5(x) = -\frac{x^3}{6} + x$$



Theorem (B. Taylor, 1715). Let $n \in \mathrm{N}$ and $f: (a, b) \to \mathrm{R}$ have continuous derivatives $f^{(k)}$ for $k = 0, 1, \ldots, n$ on the interval (a, b) and the derivative $f^{(n+1)}$ exist on the interval (a, b). If then $x_0 \in (a, b)$, then for every $x \in (a, b)$ there is a point c lying between x_0 and x and it holds:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.$$

https://en.wikipedia.org/wiki/Taylor%27s_theorem

Proof. Let the points x, x_0 be arbitrarily chosen in the interval (a, b). Furthermore, let J denote a closed interval whose endpoints are x_0 and x. Then for each $t \in J$ we put:

$$F(t) := f(x) - f(t) - f'(t)(x - t) - \frac{f''(t)}{2!}(x - t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x - t)^n.$$

Then:

$$F'(t) = -\frac{f^{(n+1)}(t)}{n!}(x - t)^n.$$

Now let's define the function G on the interval J by the formula:

$$G(t) := F(t) - \left(\frac{x - t}{x - x_0}\right)^{n+1} F(x_0), \quad t \in J.$$

Furthermore, it follows from the definition of the function $G : J \to \mathrm{R}$ that it fulfills the assumptions of Rolle's theorem. Therefore, there is a $c \in int(J)$ point for which:

$$0 = G'(c) = F'(c) + (n+1)\frac{(x-c)^n}{(x-x_0)^{n+1}}F(x_0).$$

It follows from the last statement:

$$\begin{aligned} F(x_0) &= -\frac{1}{n+1}\frac{(x-x_0)^{n+1}}{(x-c)^n}F'(c) \\ &= \frac{1}{n+1}\frac{(x-x_0)^{n+1}}{(x-c)^n}\frac{(x-c)^n}{n!}f^{(n+1)}(c) \\ &= \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}. \end{aligned}$$

This already implies a proven statement. ◻

For the so-called **remainder in Lagrangian form** we introduce the notation:

$$R_n(x) := f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

Theorem (Remainder in integral form). Suppose that the function f has on the interval $I = \langle a, b \rangle$ the derivative $f^{(k)}$ for $k = 1, 2, \dots, n+1$ and the derivative $f^{(n+1)}$ has a Riemann integral on I. Then:

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(b-a)^n + R_n,$$

where R_n is a remainder of the form:

$$R_n = \frac{1}{n!}\int_a^b f^{(n+1)}(t) \cdot (b-t)^n \, dt.$$

Proof. If we apply per partes integration to the integral of (b), we get:

$$\begin{aligned} R_n &= \frac{1}{n!}\int_a^b f^{(n+1)}(t) \cdot (b-t)^n \, dt \\ &= \frac{1}{n!}\left[f^{(n)}(t) \cdot (b-t)^n\right]_a^b + \frac{1}{(n-1)!}\int_a^b f^{(n)}(t) \cdot (b-t)^{n-1} \, dt \\ &= -\frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{1}{(n-1)!}\int_a^b f^{(n)}(t) \cdot (b-t)^{n-1} \, dt \end{aligned}$$

Using integration by the per partes method again, we then get the proof of equality (a). ◻

Example. Let's find an approximation of the function $f(x) = \sqrt[3]{1+x}$ using the Taylor polynomial $P_2(x)$ at the point $x_0 = 0$ and where $x > -1$.

Solution.

$$f'(x) = \frac{1}{3}\sqrt[-2/3]{1+x} \implies f'(0) = \frac{1}{3}$$
$$f''(x) = -\frac{2}{9}\sqrt[-5/3]{1+x} \implies f''(0) = -\frac{2}{9}.$$

After that

$$f(x) = P_2(x) + R_2(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + R_2(x),$$

where $R_2(x) = \frac{1}{3!}f'''(c)x^3 = \frac{5}{81}(1+c)^{-8/3}x^3$ for some c point lying between 0 and x.

Now let's estimate the size of the $R_2(x)$ residual for $x = 0.3$. One can then calculate $P_2(0.3) = 1.09$ which is an approximation of the function value $f(0.3) = \sqrt[3]{1.3}$. Since it is now $c > 0$, the guess: $(1+c)^{-8/3} < 1$ and therefore

$$R_2(0.3) < \frac{5}{81}0.3^3 = \frac{1}{600}.$$

Example. Let's calculate the value of e to 5 decimal places using a Taylor polynomial.

Solution. Consider the function $f(x) = e^x$. Then we find the Taylor polynomial $P_n(x)$ at the point $x_0 = 0$. Since $f^{(k)}(x) = e^x$ for $k = 0, 1, 2, \ldots$, it can be found that

$$f^{(k)}(0) = 1 \quad \forall k \in \mathrm{N}.$$

Then we have:

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}.$$

Now for $x = 1$, let's look for n such that $|P_n(1) - e| < 10^{-5}$. I mean

$$|R_n(1)| < 10^{-5}.$$

If we use the Lagrange residue, we get:

$$R_n(1) = \frac{e^c}{(n+1)!},$$

where $0 < c < 1$. Since $e^c < 3$, let's look for n such that $\frac{3}{(n+1)!} < 10^{-5}$. From there it can be seen that $n = 8$.

In [ ]:
```
# napišme funkci, která vrací hodnotu výrazu 3 / (n + 1)!
import sympy as sp
def f(n):
    return 3 / sp.factorial(n + 1)

n = 1
while True:
    if f(n) < 10**(-5):
        break
    n += 1

print("n = ", n)
```

```
n =  8
```

I mean

$$e \approx P_8(1) = 1 + 1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \frac{1}{4\,!} + \frac{1}{5\,!} + \frac{1}{6\,!} + \frac{1}{7\,!} + \frac{1}{8\,!} = 2.71828.$$

$$e = \lim_{n \to \infty} P_n(1) = \sum_{k=0}^{\infty} \frac{1}{k\,!}.$$

Example. Let's show using Taylor's theorem that $e^x > 1 + x$ holds for $x > 0$. From here we derive the inequality

$$e^{\pi} > \pi^{e}.$$

$$e = \lim_{n \to \infty} P_n(1) = \sum_{k=0}^{\infty} \frac{1}{k\,!}.$$

Solution. As we already know from the previous example, the following applies:

$$e^x = 1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \cdots + \frac{x^n}{n\,!} + R_n(x),$$

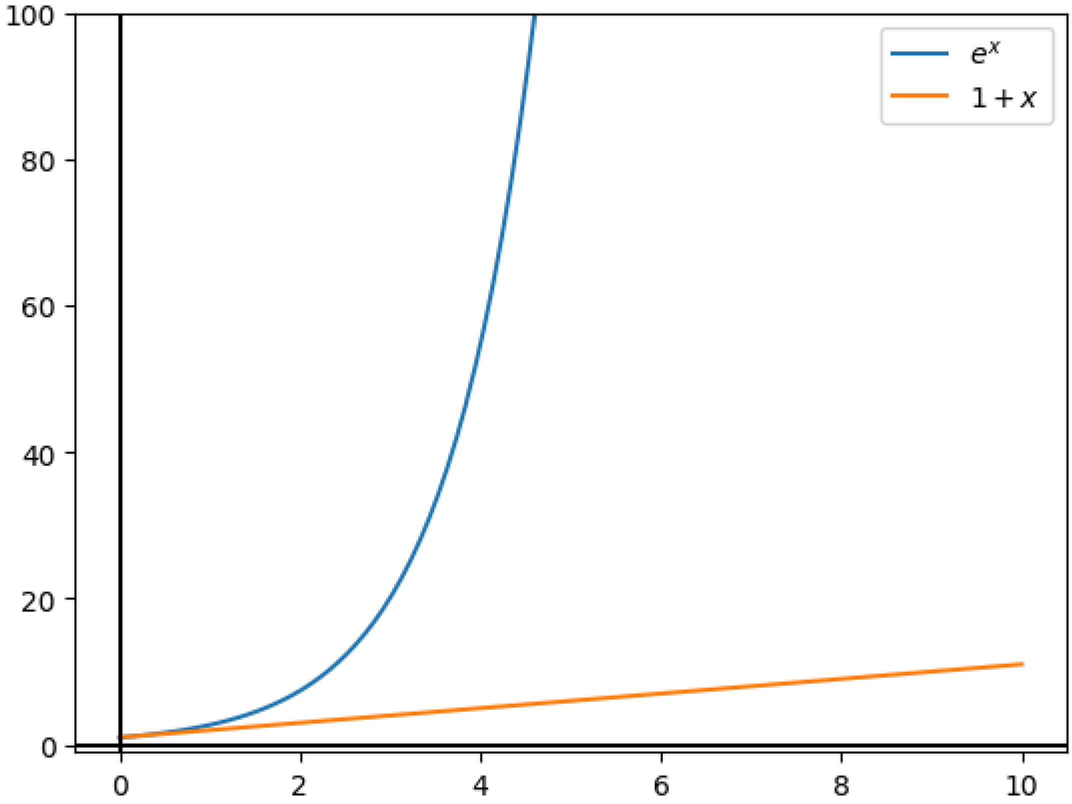
where $R_n(x) = \frac{e^c}{(n+1)\,!}$ for some $0 < c < x$. It follows from there:

$$e^x = P_1(x) + R_1(x) = 1 + x + R_1(x) = 1 + x + \frac{e^c}{2\,!}x^2 > 1 + x,$$

for it is $\frac{e^c}{2\,!}x^2 > 0$.

In [ ]:
```
# vytvořme nyní graf funkce y = e^x a y = 1 + x
import sympy as sp
import numpy as np
import matplotlib.pyplot as plt
from IPython.display import display, Math, Latex

sp.init_printing(use_latex='mathjax')
```

```python
x = sp.Symbol('x')
f = sp.exp(x)
g = 1 + x

a = 0
b = 10

x_values = np.linspace(a, b, 100)  # vytvoříme 100 hodnot x v intervalu [a,b]

y1 = [f.subs(x, x1) for x1 in x_values]  # vypočítáme hodnoty funkce f(x) pro ka
y1 = np.array(y1)  # převedeme na numpy pole

y2 = [g.subs(x, x1) for x1 in x_values]
y2 = np.array(y2) # převedeme na numpy pole

plt.plot(x_values, y1, label=r"$e^x$")    # vykreslíme graf funkce f(x)
plt.plot(x_values, y2, label=r"$1 + x$")   # vykreslíme graf funkce g(x)
plt.legend()    # zobrazíme legendu
plt.ylim(-1, 100) # nastavíme rozsah osy ybb
# přidejme plot obou os
plt.axhline(y=0, color='k')
plt.axvline(x=0, color='k')

plt.show() # zobrazíme graf
```



Because as we know $\pi > e$, we have $x = \pi/e - 1 > 0$. I mean

$$e^{\pi/e-1} > 1 + (\pi/e - 1) = \pi/e.$$

This implies that $e^{\pi/e} > (\pi/e)e = \pi$. Hence $e^{\pi} > \pi^{e}$. $\square$