Infinite number series

Basic definition

Definition 1. Consider the number sequence $X = (a_n)_{n \in \mathbb{N}}$. Then the expression

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \tag{1}$$

we will call **infinite number series**. We will use the symbol to indicate the row

$$\sum_{n=1}^{\infty} a_n, \text{ nebo zkráceně } \sum a_n.$$

The so-called **sequence of partial sums** is associated with the infinite number series $\sum a_n$

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &\vdots \end{aligned}$$

Definition. Consider the infinite number series $\sum_{n=1}^{\infty} a_n$, let s_n denote the so-called **nth partial sum** given by the relation

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

If the sequence of partial sums s_n is **convergent** and $s = \lim_{n \to \infty} s_n$, then we call the series $\sum_{n=1}^{\infty} a_n$ **convergent** and the limit s is called the **sum of the series**. If the sequence of partial sums is divergent, then we call the series **divergent**.

If the limit of $\lim_{n \to \infty} s_n$ is equal to $\pm\infty$, then we say that the series **diverges to** $\pm\infty$**.**

If the $\lim_{n \to \infty} s_n$ limit does not exist, then we say that the series **oscillates**.

Example. Consider the sequence of numbers $a_n = \frac{1}{n(n+1)}$. Let's find out if the series $\sum_{n=1}^{\infty} a_n$ is convergent and, if necessary, calculate its sum.

Solution. We determine whether the sequence of partial sums s_n is convergent. For $n \in \mathbb{N}$ we have

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}.$$

Každý člen $\frac{1}{i(i+1)}$ lze rozložit na rozdíl dvou parciálních zlomků:

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}.$$

Potom máme:

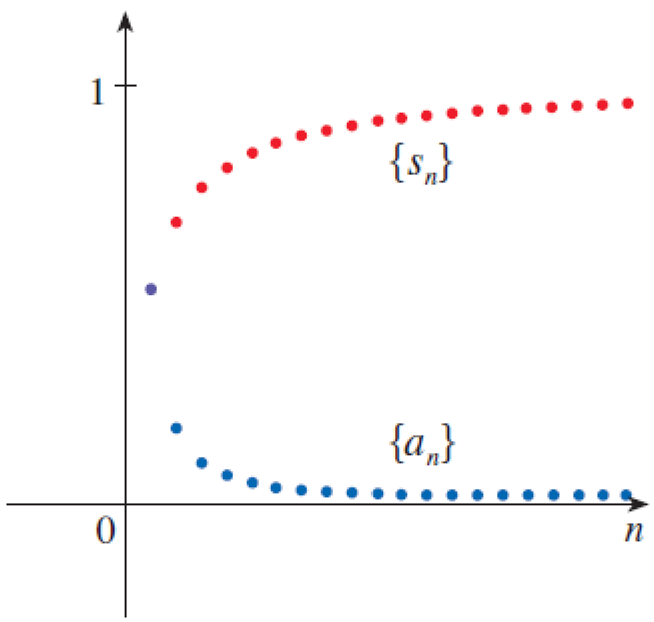
$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$$
$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= 1 - \frac{1}{n+1}$$

I mean

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$$

Odtud vyplývá pro součet:

$$s = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$



Suma geometrické řady

An important example of an irregular geometric series is the so-called **geometric series**:

$$\sum_{n=1}^{\infty} aq^{n-1} = a + aq + aq^2 + aq^3 + \ldots + aq^{n-1} + \ldots.$$

The q parameter is the so-called **quotient** of a geometric series. If $a \neq 0$ and $q = 1$ then $s_n = a + a + a + \ldots = na$ and $s_n \to \pm\infty$. For in this case the sequence of partial sums is divergent, the geometric series is also divergent. If $a = 0$ then $s_n = 0 + 0 + 0 + \ldots = 0$ and $s_n \to 0$.

Je-li $q \neq 1$, pak

$$\begin{aligned} s_n &= a + aq + aq^2 + aq^3 + \ldots + aq^{n-1} \\ qs_n &= 0 + aq + aq^2 + aq^3 + \ldots + aq^{n-1} + aq^n \end{aligned}$$

If we subtract the second equation from the first equation, we get:

$$s_n - qs_n = a - aq^n = a(1 - q^n).$$

s_n can now be expressed in closed form:

$$\boxed{s_n = a\frac{1-q^n}{1-q}.}$$

If $|q| < 1$ then $q^n \to 0$ for $n \to \infty$ and so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-q^n)}{1-q} = \frac{a}{1-q}.$$

Therefore, if $|q| < 1$, then the geometric series is convergent and its sum holds:

$$\boxed{s = \sum_{n=1}^{\infty} aq^{n-1} = \frac{a}{1-q}.}$$

Example. Let's find the sum of a geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \ldots.$$

Solution. The first term is $a = 5$, the quotient is $q = -\frac{2}{3}$. Since $|q| < 1$, the geometric series is convergent and its sum is

$$s = \frac{a}{1-q} = \frac{5}{1-(-\frac{2}{3})} = \frac{5}{\frac{5}{3}} = 3.$$

Example. Let's investigate the convergence or divergence of the series

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}.$$

Solution. Let's express the nth term of the series in the form aq^{n-1}:

$$2^{2n}3^{1-n} = (2^2)^n 3^{-(n-1)} = \frac{4^n}{3^{n-1}} = 4\left(\frac{4}{3}\right)^{n-1}.$$

I mean

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} 4\left(\frac{4}{3}\right)^{n-1}.$$

The second method to determine the a and q parameters of a geometric series is to break down the sum:

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = 4 + \frac{16}{3} + \frac{64}{9} + \frac{256}{27} + \dots.$$

It follows from the above that $a = 4$ and $q = \frac{4}{3}$. Because $|q| > 1$, a geometric series is divergent.

Harmonická řada

We call the **harmonic series** a series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots.$$

This series is divergent.

Proof. Consider a selected sequence of partial sums: $s_2, s_4, s_6, \dots$. Then

$$s_2 = 1 + \tfrac{1}{2}$$

$$s_4 = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) > 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = 1 + \tfrac{2}{2}$$

$$\begin{aligned} s_8 &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) \\ &> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2} \end{aligned}$$

$$\begin{aligned} s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right) \\ &> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right) \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{4}{2} \end{aligned}$$

Obecně lze ukázat, že

$$s_{2^n} > 1 + \frac{n}{2}, \quad n = 1, 2, 3, \ldots$$

From here it flows: $s_{2^n} \to \infty$ for $n \to \infty$. So $s_n \to \infty$ for $n \to \infty$. The series is therefore divergent.

Základní vlastnosti konvergentních řad

Theorem. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then:

$$\lim_{n \to \infty} a_n = 0.$$

Proof. For $n \geq 2$: $a_n = s_n - s_{n-1}$. From the convergence of the series $\sum_{n=1}^{\infty} a_n$ follows: $s_n \to s$. It follows: $s_n - s_{n-1} \to s - s = 0$. So $\lim_{n \to \infty} a_n = 0.\square$

Example. Consider an infinite series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} = 1 - 1 + 1 - 1 + \ldots.$$

Note. The previous theorem cannot be reversed as can be demonstrated by the example of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. This series diverges, but $\lim_{n \to \infty} \frac{1}{n} = 0$. The previous theorem is therefore only a **necessary condition** for the convergence of the series.

Example. Let's prove the divergence of a series

$$\sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4}.$$

Solution.

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2}{5n^2 + 4} = \lim_{n \to \infty} \frac{1}{5 + \frac{4}{n^2}} = \frac{1}{5} \neq 0.$$

Because the necessary condition of convergence is not met for this series, the series is therefore divergent.$\square$

Theorem. Assume that the series $\sum a_n$ and $\sum b_n$ are convergent. Then the series $\sum c a_n$ (where c is a constant), $\sum (a_n + b_n)$, $\sum (a_n - b_n)$ also converge and hold:

(i) $\sum c a_n = c \sum a_n$,

(ii) $\sum (a_n + b_n) = \sum a_n + \sum b_n$,

(iii) $\sum (a_n - b_n) = \sum a_n - \sum b_n$.

Proof. Let's prove, for example, part (ii). The other two parts prove similarly. Let's put:

$$s_n = \sum_{i=1}^{n} a_i, \quad s = \sum_{n=1}^{\infty} a_n, \quad t_n = \sum_{i=1}^{n} b_i, \quad t = \sum_{n=1}^{\infty} b_n.$$

If then $u_n = \sum_{i=1}^{n}(a_i + b_i)$ then $u_n = s_n + t_n$. From the convergence of the series $\sum a_n$ and $\sum b_n$ follows: $s_n + t_n \to s + t$. So $u_n \to s + t$. So $\sum(a_n + b_n) = \sum a_n + \sum b_n$.$\square$

Example. Let's find the sum of a series

$$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right).$$

Solution. First we determine the sums of the series

$$\sum_{n=1}^{\infty} \frac{3}{n(n+1)}, \quad \sum_{n=1}^{\infty} \frac{1}{2^n}.$$

$$\sum_{n=1}^{\infty} \frac{3}{n(n+1)} = 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 3 \cdot 1 = 3.$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

Now by the previous theorem, part (ii), the given series converges and its sum is

$$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right) = 3 + 1 = 4.$$

Integral series convergence test

Sentence. Let $\{a_n\}$ be a decreasing nonnegative sequence, $n \geq 1$, and $f(x)$ be a decreasing, continuous, nonnegative function on the interval $[1, \infty)$ such that $f(n) = a_n$. Then the number series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ converges.

Here we have three simple examples to which we can apply the Integral Test of Convergence of a Number Series:

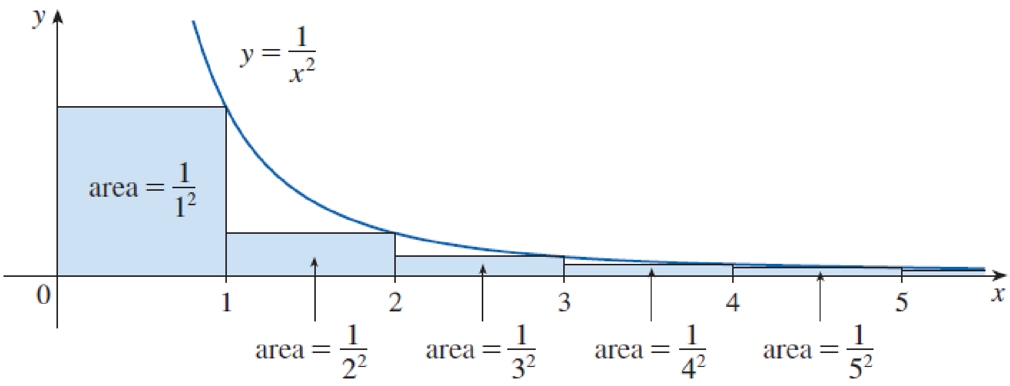
Example 1: Let's verify the convergence of the number series $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Example 2: Let's verify the convergence of the number series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

Example 3: Let's verify the convergence of the number series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.

In each example, we first find a corresponding decreasing, continuous, non-negative function $f(x)$ that satisfies the conditions of the Integral Test. We then calculate the improper integral $\int_1^\infty f(x), dx$ and determine whether it converges or diverges. Based on the result of the integral, we can then determine whether the number series converges or diverges.

Let's first comment on Example 1. Notice the following image:



It is obvious that the sum of the contents of the individual rectangles is equal to the sum:

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Furthermore, it is clear that for each partial sum of this number series, the upper estimate holds:

$$s_k = \sum_{n=1}^{k} \frac{1}{n^2} \leq \frac{1}{1^2} + \int_1^\infty \frac{1}{x^2} \, \mathrm{d}x = 2.$$

Thus, the sequence of partial sums $\{s_k\}$ is an increasing and bounded sequence from above. Hence the sequence $\{s_k\}$ is convergent. This means that the given series is convergent.☐

Note. L. Euler proved that it holds for the sum of this series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

end 26.3.

Let us solve Example 3 as a demonstration of the application of the integral criterion. We want to verify the convergence of the number series $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2}$. First, we find the corresponding decreasing, continuous, non-negative function $f(x)$ that satisfies the conditions of the Integral Test:

$f(x) = \frac{1}{x(\ln x)^2}$ where $x \geq 2$ (for $x = 1$ the function is not defined because $\ln 1 = 0$).

Now we calculate the step integral $\int_2^\infty f(x), dx$:

$\int_2^\infty \frac{1}{x(\ln x)^2}, dx.$

To solve this integral, we use the substitution $u = \ln x$, which means that $du = \frac{1}{x} dx$. We will also adjust the limits of integration:

If $x = 2$ then $u = \ln 2$. If $x \to \infty$ then $u \to \infty$. The integral then looks like this:

$\int_{\ln 2}^\infty \frac{1}{u^2}, du.$

This integral is now simpler and we can solve it using the rule for integrating power functions:

$\int u^n, du = \frac{u^{n+1}}{n+1} + C$ where $n \neq -1$.

We apply this rule to our integral:

$\int_{\ln 2}^\infty \frac{1}{u^2}, du = \left[-\frac{1}{u}\right]_{\ln 2}^\infty.$

Now we find the value of the integral in the limiting values:

$\lim_{u\to\infty} \left(-\frac{1}{u}\right) - \left(-\frac{1}{\ln 2}\right) = 0 - (-\frac{1}{\ln 2}) = \frac{1}{\ln 2}.$

Since the improper integral of $\int_2^\infty f(x), dx$ converges, the number series $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ also converges by the Integral Convergence Test.

Theorem. The harmonic series $\sum_{n=1}^\infty \frac{1}{n}$ is a divergent series.

Proof. To prove the divergence, we will use the Integral Convergence Test. First, we find the corresponding decreasing, continuous, non-negative function $f(x)$ that satisfies the conditions of the Integral Test:

$$f(x) = \frac{1}{x}, \quad x \geq 1.$$

Now we calculate the step integral $\int_1^\infty f(x), dx$:

$$\int_1^\infty \frac{1}{x}, dx = [\ln x]_1^\infty = \lim_{t\to\infty} [(\ln t) - (\ln 1)] = \infty - 0 = \infty.$$

Since the $\int_1^\infty f(x), dx$ step integral diverges, the $\sum_{n=1}^\infty \frac{1}{n}$ number series also diverges by the Integral Convergence Test.□

Example. Use the integral criterion to determine whether the series $\sum_{n=1}^\infty \frac{\ln n}{n}$ is convergent or divergent.

Solution. First, we find the corresponding decreasing, continuous, nonnegative function $f(x)$ that satisfies the conditions of the Integral Test:

$$f(x) = \frac{\ln x}{x}, \quad x \geq 1.$$

$$\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x}, dx$$
$$= \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t$$
$$= \lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty.$$

Clearly, for $x > 1$, $f(x) > 0$ and f is a continuous function here. However, it is not obvious how it is with the monotonicity of $f(x)$. So let's try to calculate the $f'(x)$ derivative:

$$f'(x) = \frac{1}{x^2} - \frac{\ln x}{x^2} = \frac{1-\ln x}{x^2}.$$

From this expression it is clear that $f'(x) < 0$ for $x > 1$. So $f(x)$ is a decreasing function. Now we calculate the step integral $\int_1^\infty f(x), dx$:

Since the given improper integral diverges, the number series $\sum_{n=1}^\infty \frac{\ln n}{n}$ also diverges by the Integral Convergence Test.☐

Theorem (Estimating the rest of the series). If $f(k) = a_k$, where f is a continuous, positive and decreasing function for $x \geq n$ and the series $\sum a_n$ is convergent, then if $R_n = s - s_n$, then

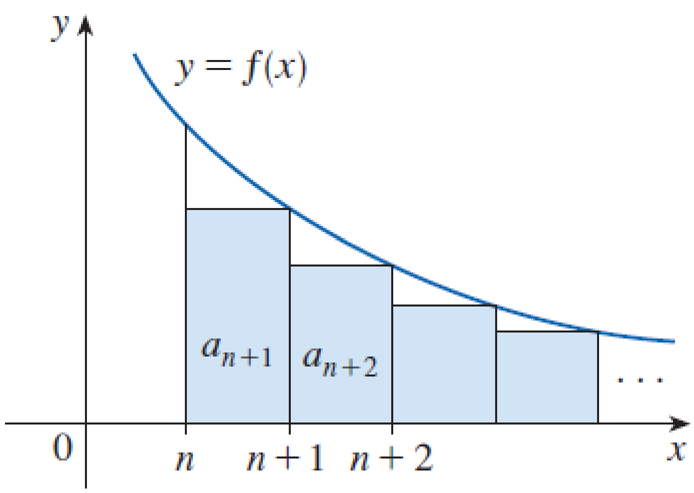
$$\int_{n+1}^\infty f(x)dx \leq R_n \leq \int_n^\infty f(x)dx.$$

Proof. Here it is

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

Then as the figure below shows:

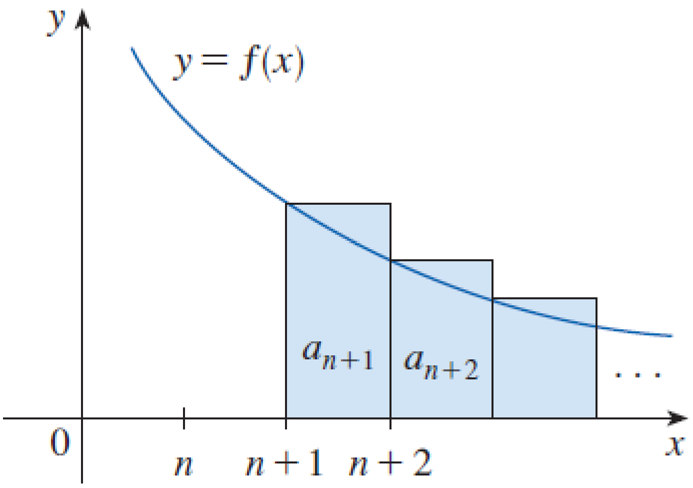
$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \leq \int_n^\infty f(x), dx.$$



In addition, the following applies:

$$R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^{\infty} f(x)dx.$$

See the following image.



Example. As can be easily found using the integral criterion, a series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

is convergent.

(a) Let us approximate the sum of this series by the sum of its first ten terms:

$$\sum_{n=1}^{10} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \cdots + \frac{1}{10^3}.$$

Then let's estimate the error that occurs when using this approximation.

(b) How many terms of the series need to be used so that the error is less than 5×10^{-4} ?

Solution. We will calculate the improper integral $\int_n^\infty \frac{1}{x^3}\,dx$ to solve both parts.

$$\int_n^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2} + \frac{1}{2n^2}\right) = \frac{1}{2n^2}.$$

(a) Let us approximate the sum of this series by the sum of its first ten terms:

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx s_{10} = \sum_{n=1}^{10} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \cdots + \frac{1}{10^3} \approx 1.1975.$$

Then, by the residual estimation theorem, estimate the error that occurs when using this approximation:

$$R_{10} \leq \int_{10}^{\infty} \frac{1}{x^3}\, dx = \frac{1}{200} = 0.005.$$

(b) To achieve 5×10^{-4} precision, this means that we want for n large enough to hold:

$$R_n \leq 5 \times 10^{-4}.$$

Because it is

$$R_n \leq \int_{n}^{\infty} \frac{1}{x^3}\, dx = \frac{1}{2n^2},$$

we need it to hold:

$$\frac{1}{2n^2} \leq 5 \times 10^{-4}.$$

In [ ]:
```
# Napišme skript, kde najdeme nejmenší n
# takové, že bude platit podmínka: 1 / (2*n**2) < 5*10**(-4)
import sympy as sp
n = 1
while True:
    if 1 / (2*n**2) <= 5*10**(-4):
        break
    n += 1

print("n = ", n)
```

```
n =  32
```

Therefore, according to the numerical experiment above, $n = 32$ series members need to be used to make the error smaller than 5×10^{-4}.$\square$

Note. If we go back to estimating the rest of the series

$$\int_{n+1}^{\infty} f(x), dx \leq R_n \leq \int_{n}^{\infty} f(x)\, dx,$$

then using the relation $s_n + R_n = s$ we easily get from the previous estimates:

$$\boxed{s_n + \int_{n+1}^{\infty} f(x)\, dx \leq s \leq s_n + \int_{n}^{\infty} f(x)\, dx.} \qquad (*)$$

Example. Let's use the estimates (*) and the partial sum s_{10} of the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ to estimate the sum of this series.

Solution. According to the above estimate (*):

$$s_{10} + \int_{11}^{\infty} \frac{1}{x^3}\, dx \leq s \leq s_{10} + \int_{10}^{\infty} \frac{1}{x^3}\, dx.$$

Jak již víme,

$$\int_{n}^{\infty} \frac{1}{x^3}\, dx = \frac{1}{2n^2},$$

tudíž platí:

$$s_{10} + \frac{1}{2(11)^2} \leq s \leq s_{10} + \frac{1}{2(10)^2}.$$

Užitím hodnoty $s_{10} \approx 1.197532$, dostaneme:

$$1.201664 \leq s \leq 1.202532.$$

For example, if we approximate the sum using the center of this interval, we get:

$$s \approx \frac{1.201664 + 1.202532}{2} = 1.2021,$$

with an error less than half the length of the interval, i.e. 0.0005. So we see that to achieve an error smaller than 5×10^{-4} it is sufficient to use only ten terms of the series. $\square$

Srovnávací kritérium

Theorem. Assume that the series $\sum a_n$ and $\sum b_n$ are series with nonnegative terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ holds for every n, then $\sum a_n$ is convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ holds for every n, then $\sum a_n$ is divergent.

Proof. Do the proof yourself.$\square$

Theorem. (i) The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent to $p > 1$.

(ii) The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent for $p \leq 1$.

(iii) The series $\sum_{n=1}^{\infty} aq^{n-1}$ is convergent if and only if $|q| < 1$.

Example. Determine whether the series $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ is convergent or divergent.

Solution. According to the comparison criterion theorem, the series is convergent. Let's compare the given series with the $\sum_{n=1}^{\infty} \frac{5}{2n^2}$ series, which is convergent. It is valid

$$\frac{5}{2n^2 + 4n + 3} \leq \frac{5}{2n^2},$$

for each n. The inequality is apparently valid because the denominator on the left side is greater than the denominator on the right side. Next we know

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Hence, using the comparison criterion theorem (part (i)), it follows that the series $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ is convergent.□

Note. Conditions $a_n \leq b_n$ or $a_n \geq b_n$ verify for $n \geq N$ where N is a sufficiently large fixed natural number.

Example. Determine whether the series $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ is convergent or divergent.

Solution. Here we easily find that $\ln k > 1$ for every $k \geq 3$. So:

$$\frac{\ln k}{k} > \frac{1}{k}, \quad k \geq 3.$$

Since, as we know, the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent, so is the series $\sum_{k=1}^{\infty} \frac{\ln k}{k}$ divergent.□

Alternating series and the concept of absolute convergence

By **alternating series** or **series with alternating signs** we mean a series whose members alternate signs. For example, these are the series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = \ln 2,$$

or

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \frac{7}{8} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}.$$

From the given examples, it is clear that a_n applies to the members of these series

$$a_n = (-1)^{n-1} b_n, \text{ resp. } a_n = (-1)^n b_n,$$

where $b_n > 0.$

Theorem (Leibniz Criterion). Series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots \quad (b_n > 0)$$

meeting conditions:

(i) $b_{n+1} \leq b_n, \quad \forall n \in \mathbb{N}$

(ii) $\lim_{n \to \infty} b_n = 0$

is convergent.

Example. Consider an alternating series:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}.$$

Let us prove that this (Leibnizian) series is convergent.

(i) $b_{n+1} \leq b_n$ since $\frac{1}{n+1} \leq \frac{1}{n}$ for every $n \geq 1.$

(ii) $\lim_{n\to\infty} b_n = 0$ because $\lim_{n\to\infty} \frac{1}{n} = 0.$

Thus, the given series satisfies conditions (i) and (ii) and is convergent.□

Example. Consider an alternating series:

$$\sum_{n=1}^{\infty}\frac{(-1)^n 3n}{4n-1}.$$

Now $b_n = \frac{3n}{4n-1}$. Then

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{3n}{4n-1} = \lim_{n\to\infty}\frac{3}{4}\cdot\frac{1}{1-\frac{1}{4n}} = \frac{3}{4}.$$

Therefore, Leibni's criterion cannot be applied. Next, let's verify whether the necessary convergence condition $\lim_{n\to\infty} a_n = 0.$ is satisfied

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^n\frac{3n}{4n-1}.$$

However, this limit does not exist. So the given series is divergent.□

Pojem absolutní konvergence

Given the series $\sum a_n$, then consider a series of absolute values:

$$\sum_{n=1}^{\infty}|a_n| = |a_1|+|a_2|+|a_3|+\cdots.$$

Definition. A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Example. Consider an alternating series:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} = 1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\frac{1}{36}+\cdots.$$

This series is absolutely convergent, since the corresponding series of absolute values

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{n^2}\right| = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots$$

converges. So the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ is absolutely convergent.□

Theorem. If the series $\sum a_n$ is absolutely convergent, then the series is convergent.

Example. Consider the series:

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \frac{\cos 4}{4^2} + \cdots.$$

Since for every $n \in \mathbb{N}$ holds

$$\left|\frac{\cos n}{n^2}\right| \leq \frac{1}{n^2},$$

so according to the comparison test, we can say that the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is absolutely convergent. So the series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent.□

Podílové a odmocninové kritérium

Theorem. (i) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum a_n$ is absolutely convergent (and also convergent).

(ii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, then the series $\sum a_n$ is convergent or divergent and no definite conclusions can be drawn about the convergence of the series $\sum a_n$.

Example. Let's examine the convergence of the series:

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}.$$

Solution. Let's use the quotient criterion where $a_n = (-1)^n \frac{n^3}{3^n}$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$
$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1 + \frac{1}{n}\right)^3 \to \frac{1}{3} < 1.$$

The quotient criterion then implies that the series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ is absolutely convergent.□

Example. Let's examine the convergence of the series:

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}.$$

Solution. Let's use the quotient criterion where $a_n = \frac{n^n}{n!}$:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)(n+1)^n}{(n+1)!n!} \cdot \frac{n!}{n^n}$$
$$= \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \to e, \quad \text{pro } n \to \infty.$$

Because $e > 1$,, according to the ratio criterion, the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent.□

Note. Divergence also follows from the so-called necessary condition, since:

$$a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot \dots \cdot n}{1 \cdot 2 \cdot \dots \cdot n} \geq n.$$

So the sequence a_n does not converge to 0 for $n \to \infty$. Hence the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ is divergent.

Example. Let's examine the convergence of the $\sum_{n=1}^{\infty} \frac{1}{n}$ series using the quotient criterion. Now $a_n = \frac{1}{n}$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1/(n+1)}{1/n} = \frac{n+1}{n} \to 1, \quad \text{pro } n \to \infty.$$

Therefore, the ratio criterion cannot be used here.□

Theorem. (i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum a_n$ is absolutely convergent (and also convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then the convergence or divergence of the series $\sum a_n$ cannot be decided according to the root criterion.

Example. Let's examine the convergence of the $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ series using the root criterion.

Solution. Here

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n.$$

$$\sqrt[n]{|a_n|} = \left(\frac{2n+3}{3n+2}\right)^{\frac{n}{n}} = \left(\frac{2n+3}{3n+2}\right)^1 = \frac{2n+3}{3n+2} \to \frac{2}{3} < 1.$$

The series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$ is therefore absolutely convergent.□

Example. Let's investigate the convergence of a series

$$\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^n.$$

Solution. Here

$$a_n=\left(\frac{n}{n+1}\right)^n.$$

It flows from there

$$\sqrt[n]{|a_n|}=\frac{n}{n+1}\to 1 \text{ pro } n\to\infty.$$

Since the last limit is equal to 1, the square root criterion cannot be used to examine convergence or divergence. Next

$$a_n=\left(\frac{n}{n+1}\right)^n=\frac{1}{\left(\frac{n+1}{n}\right)^n}\to\frac{1}{e} \ \text{ pro } n\to\infty.$$

Since this limit is different from 0, the necessary condition for convergence of the series is not fulfilled $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^n$. Therefore, the given series is divergent.□