

Comprehensive Examination Guide: Infinite Series and Integral Calculus

Examiner's Preface

This guide is designed for candidates prepared to demonstrate a rigorous understanding of the transition from discrete summation to continuous integration. As an examiner, I require more than the recitation of formulas; you must demonstrate the ability to derive results from first principles, justify the necessity of theoretical conditions, and provide precise error estimates. Mathematical rigor is the standard.

1 Fundamental Concepts of Infinite Number Series

Analyze the core definitions that bridge the gap between discrete sequences and infinite sums. This section establishes the theoretical groundwork for understanding how an infinite sum can yield a finite value.

Examination Question 1.1

Define the sequence of partial sums $\{s_n\}$ for a given sequence $\{a_n\}$. Under what precise mathematical condition is the infinite series $\sum_{n=1}^{\infty} a_n$ considered convergent?

Examination Question 1.2

Distinguish between a series that is "divergent to $\pm\infty$ " and one that "oscillates." Provide the limit conditions for $\{s_n\}$ in both cases.

Examination Question 1.3

Using the example $a_n = \frac{1}{n(n+1)}$, use the method of partial fractions to show that $s_n = 1 - \frac{1}{n+1}$. Use this result to determine the sum of the series.

Examiner's Key Hints

- **Partial Sums:** Recall $s_n = \sum_{i=1}^n a_i$. Convergence is defined solely by the existence of a finite limit $s = \lim_{n \rightarrow \infty} s_n$.
- **Oscillation:** If $\lim_{n \rightarrow \infty} s_n$ does not exist and is not $\pm\infty$, the series oscillates.
- **Telescoping:** Decompose a_n into $\frac{1}{n} - \frac{1}{n+1}$ and observe the internal cancellations.

2 Geometric and Harmonic Series: Divergence and Convergence

Evaluate the specific parameters that dictate the behavior of geometric and harmonic series. These serve as the primary benchmarks for the comparison criteria used in advanced calculus.

Examination Question 2.1

Derive the sum of a geometric series $\sum_{n=1}^{\infty} aq^{n-1}$. You must show the algebraic step involving $s_n - qs_n$ to find the closed form of s_n . State the domain of the quotient q for which the series converges to $s = \frac{a}{1-q}$.

Examination Question 2.2

The Harmonic Series $\sum \frac{1}{n}$ is a fundamental divergent series. Replicate the proof of divergence by providing estimates for $s_2, s_4, s_8,$ and s_{16} . Show how the general estimate $s_{2^n} > 1 + \frac{n}{2}$ leads to the conclusion of divergence.

Examination Question 2.3: Calculation Challenge

Analyze the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$. Identify the first term a and the quotient q . Based on these parameters, justify whether the series converges or diverges.

Examination Question 2.4

State the "Necessary Condition for Convergence." Explain why $\lim_{n \rightarrow \infty} a_n = 0$ is necessary but not sufficient for convergence, using the Harmonic Series as your counter-example.

Examiner's Key Hints

- **Geometric Derivation:** Start with $s_n = a + aq + \dots + aq^{n-1}$ and subtract qs_n . Note that convergence requires $|q| < 1$.
- **Harmonic Proof:** Group terms such that $(\frac{1}{3} + \frac{1}{4}) > \frac{2}{4} = \frac{1}{2}$, and $(\frac{1}{5} + \dots + \frac{1}{8}) > \frac{4}{8} = \frac{1}{2}$.
- **Necessary Condition:** If $\sum a_n$ converges, $a_n = s_n - s_{n-1} \rightarrow s - s = 0$.

3 Comparative and Integral Convergence Tests

Analyze the tools used to determine the convergence of series that do not fit simple geometric or harmonic forms.

Examination Question 3.1: The Integral Test

Given a function $f(x)$ that is decreasing, continuous, and non-negative on $[1, \infty)$, state the relationship between $\sum a_n$ and the improper integral $\int_1^{\infty} f(x) dx$.

Examination Question 3.2: Comparison Criterion

Demonstrate the use of a majorant to prove the convergence of $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$. Which standard p -series would serve as an appropriate comparison?

Examination Question 3.3: Error Estimation

Define the remainder $R_n = s - s_n$. Provide the double inequality that bounds R_n using improper integrals of the corresponding function $f(x)$.

Examination Question 3.4: p -series

State the convergence conditions for the p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

Examiner's Key Hints

- **Majorant Strategy:** For $\frac{5}{2n^2+4n+3}$, note that $2n^2 + 4n + 3 > 2n^2$, thus $\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$.
- **Remainder Bounds:** The precise estimate is $\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx$.
- **p -series:** Convergence occurs if and only if $p > 1$.

4 Alternating Series and Absolute Convergence

Determine how sign changes affect the stability and convergence of a series.

Examination Question 4.1: Leibniz Criterion

List the two mandatory conditions for the convergence of an alternating series $\sum (-1)^{n-1} b_n$. Apply these conditions to the alternating harmonic series $\sum \frac{(-1)^{n-1}}{n}$.

Examination Question 4.2

Define Absolute Convergence. State the theorem regarding the relationship between $\sum |a_n|$ and $\sum a_n$.

Examination Question 4.3: Ratio and Root Tests

For both the Ratio ($L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$) and Root ($L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$) tests, describe the behavior of the series when $L < 1$, $L > 1$, and $L = 1$.

Examination Question 4.4

Using the example $\sum \frac{\cos n}{n^2}$, justify its convergence using the concept of absolute convergence and the comparison test.

Examiner's Key Hints

- **Leibniz:** Requires $b_{n+1} \leq b_n$ and $\lim_{n \rightarrow \infty} b_n = 0$.
- **Tests:** Remember $L = 1$ is inconclusive; a different test must be employed.
- **Cosine Series:** Note $|\frac{\cos n}{n^2}| \leq \frac{1}{n^2}$.

5 The Riemann Integral: Approximation and Definition

Evaluate the transition from finite sums to the definite integral through area approximations.

Examination Question 5.1: The Area Problem

For $y = x^2$ on $[0, 1]$, describe the construction of lower sums (L_n) and upper sums (R_n). Show how the sum of squares formula $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$ (provable by induction) is used to find the limit of these sums as $n \rightarrow \infty$.

Examination Question 5.2

Define the General Riemann Sum $\sum_{i=1}^n f(x_i^*) \Delta x$. Explain the role of "selection points" (x_i^*) and the requirement for integrability regarding these points.

Examination Question 5.3: Lebesgue Criterion

Define a set of "measure zero" precisely. State the Lebesgue Criterion for Riemann integrability in terms of the function's discontinuity points.

Examination Question 5.4

Interpret the definite integral $\int_a^b f(x) dx$ as a "signed area." If $\int_0^3 (x^3 - 6x) dx = -6.75$, what does the negative sign imply about the areas A_1 (above the axis) and A_2 (below the axis)?

Examiner's Key Hints

- **Sum of Squares:** The limit $\lim_{n \rightarrow \infty} \frac{(n+1)(2n+1)}{6n^2} = \frac{1}{3}$.
- **Measure Zero:** A set N has measure zero if for every $\epsilon > 0$, there exists a sequence of intervals I_n such that $N \subset \cup I_n$ and $\sum |I_n| < \epsilon$.
- **Signed Area:** The integral equals $A_1 - A_2$.

6 The Fundamental Theorem of Calculus and Substitution

Analyze the relationship between the primitive function and the definite integral.

Examination Question 6.1: Newton-Leibniz Formula

State the formula $\int_a^b f(x) dx = F(b) - F(a)$. You must list the three necessary conditions for this to hold. Using the function $h(x) = 1/\sqrt{x}$ on $[0, 1]$, explain why the formula fails if the function is not bounded/integrable.

Examination Question 6.2

Define a Primitive Function. Prove that if $F(x)$ is a primitive to $f(x)$, then $F(x) + C$ constitutes the general form of all primitives.

Examination Question 6.3: Integration by Substitution

Demonstrate the substitution method for $\int x^3 \cos(x^4 + 2) dx$. Show the transformation $u = g(x)$ and the resulting differential du .

Examination Question 6.4: Variable Upper Limit

If $g(x) = \int_a^x f(t) dt$, find $g'(x)$ for a continuous f . Apply this to $g(x) = \int_0^x \sqrt{1+t^2} dt$.

Examiner's Key Hints

- **NL Prerequisites:** F must be continuous on $[a, b]$, $F' = f$ on (a, b) , and f must be Riemann integrable on $[a, b]$.
- **Substitution:** Let $u = x^4 + 2$, then $du = 4x^3 dx$.
- **Variable Limit:** By the FTC, the derivative of the accumulation function is the integrand evaluated at x .

Final Review: Essential Formulas

Concept	Formula / Conditions
Geometric Series Sum	$s = \frac{a}{1-q}$ for $ q < 1$
Geometric Partial Sum	$s_n = \frac{a(1-q^n)}{1-q}$ (derived via $s_n - qs_n$)
Harmonic Divergence	$s_{2^n} > 1 + \frac{n}{2}$
Necessary Condition	$\sum a_n$ converges $\implies \lim_{n \rightarrow \infty} a_n = 0$
Integral Remainder Bounds	$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx$
p-series Test	$\sum n^{-p}$ converges if $p > 1$
Leibniz Criterion	$b_{n+1} \leq b_n$ and $\lim_{n \rightarrow \infty} b_n = 0$
Ratio Test Limit (L)	$L = \lim_{n \rightarrow \infty} \left \frac{a_{n+1}}{a_n} \right $
Root Test Limit (L)	$L = \lim_{n \rightarrow \infty} \sqrt[n]{ a_n }$
Sum of Squares	$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$
Riemann Sum	$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x$
Newton-Leibniz	$\int_a^b f(x) dx = F(b) - F(a)$
Substitution Rule	$\int f(g(x))g'(x) dx = \int f(u) du$